QE Algebra (2024 Winter)

Problem 1 (11pts) Consider the dihedral group $D_{20} = \langle r, s | r^{10} = s^2 = 1, rs = sr^{-1} \rangle$. Describe the center of the group D_{20}.

Problem 2 (11pts) Let R be a commutative ring with 1. Prove the following

$$\{x \in R \mid x^n = 0 \text{ for some positive integer } n\} = \bigcap_{\mathfrak{P}\text{:prime ideal}} \mathfrak{P}.$$

Problem 3 (11pts) (1) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.
(2) Using the above statement, describe all integer triples (x, y, z) satisfying $x^2 + 2y^2 = z^2$.

Problem 4 (11pts) Let K be a field.
(1) List, by giving generators for them, the ideals of $K[x]/\langle x^3 \rangle$.
(2) List, by giving generators for them, the prime ideals of $K[x]/\langle x^3 \rangle$.
(3) List, by giving generators for them, the maximal ideals of $K[x]/\langle x^3 \rangle$.

Problem 5 (11pts) Let F be a field. Show that every finite subgroup of the multiplicative group F^* is cyclic.

Problem 6 (11pts) The general linear group $\mathrm{GL}(n, \mathbb{C})$ of degree n over $\mathbb{C}$ is the group of $n \times n$ invertible matrices over $\mathbb{C}$ with the binary operation of ordinary matrix multiplication. Find all finite abelian subgroups of $\mathrm{GL}(2, \mathbb{C})$ up to isomorphism.

Problem 7 (11pts) Let F, K and L be three fields satisfying $F \subset K \subset L$. Show that if K/F and L/K are algebraic, then L/F is algebraic.

Problem 8 (11pts) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial over $\mathbb{Q}$. Does $f(x) = 0$ have a multiple root? Prove or disprove it.

Problem 9 (11pts) Let $f(x)$ be a polynomial of degree 3 in $\mathbb{Q}[x]$. Find all the possible orders of the Galois group of $f(x)$. (For each order, you must find $f(x)$ explicitly.)