Qualifying Examination (Analysis, 01/2024)

1. (*10 points*) Let K be a continuous function on the unit square $0 \leq x, y \leq 1$ satisfying $|K(x,y)| < 1$ for all x and y. Show that there is a continuous function $f(x)$ on $[0,1]$ such that we have

$$f(x) + \int_0^1 K(x,y)f(y)dy = e^{x^2}.$$

Can there be more than one such function f?
(Hint: define some functional $T : C([0,1]) \to C([0,1])$ and show that we can use the contraction mapping principle on T.)

2. (*10 points*) Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \cdots$ with $|f_n'(x)| \leq 1$ for all n and x. Assume

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

3. (*15 points*) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ has rank n everywhere. Suppose $f^{-1}(K)$ is compact whenever K is compact. Prove that $f(\mathbb{R}^n) = \mathbb{R}^n$.
(Hint: show that $f(\mathbb{R}^n)$ is both open and closed in $\mathbb{R}^n$. Then why does this imply $f(\mathbb{R}^n) = \mathbb{R}^n$?)

4. (*15 points*) For an integer $n \geq 1$, let $g(z) = \sum_{j=0}^{n} a_j z^j$ be a complex-valued function with $a_j \in \mathbb{C}$ for each j. Evaluate the integral

$$\frac{1}{2\pi i}\int_{|z|=5} z^{n-2}|g(z)|^2 dz.$$

5. (*10 points*) Let $f(z)$ be a complex-valued holomorphic function defined on $\mathbb{C}$ such that $|f(z)| \leq |e^z|$ for each z and $f(0) = 1$. Show that

$$\int_{|z|=1} \frac{f(z)}{z} dz = 2\pi i.$$

6. (*10 points*) Prove the Schwarz Lemma: Let $\mathbb{D}(0,1) = \{z \in \mathbb{C} : |z| < 1\}$. Let $f : \mathbb{D}(0,1) \to \mathbb{D}(0,1)$ be a complex-valued holomorphic function with $f(0) = 0$. Show that $|f'(0)| \leq 1$. (Hint: Define $g(z) = f(z)/z$ for $z \neq 0$ and apply the Maximum Modulus Principle for $g(z)$.)

7. (*10 points*) Consider the differential equation

$$u'(t) + \lambda u(t) = f(t),$$

for some constant $\lambda \in \mathbb{R}^n$ and smooth function $f : [0, \infty) \to \mathbb{R}$.
(a) Find the general solutions to the equation.
(b) Show that if (the Riemann integral) $\int_0^\infty e^{\lambda s} f(s)\, ds$ is finite then there exists unique solution to the equation such that $\lim_{t\to\infty} u(t)e^{\lambda t} = 0$.

8. (*10 points*) Consider the system of differential equations

$$\begin{aligned} x' &= +y \\ y' &= -x + 2z \\ z' &= -2y \end{aligned}$$

Show that, for any given solution, $x^2(t) + y^2(t) + z^2(t)$ is constant in t.

9. (*10 points*) Let us consider the differential equation

$$x''(t) + 2x'(t) + 8x(t) = 8\cos 2t. \tag{1}$$

(a) Find the general solutions $x_h(t)$ to the homogeneous equation

$$x_h''(t) + 2x_h'(t) + 8x_h'(t) = 0.$$

(b) Let $x(t)$ be a solution to the differential equation (1). Show that

$$\lim_{t\to\infty} (x(t) - \alpha\cos(2t - \beta)) = 0,$$

for some properly chosen $\alpha \in (0, \infty)$ and $\beta \in [0, 2\pi)$.