확률및통계 2024-1학기 수학과 졸업시험

Graduation Exam **Probability & Statistics** **Spring 2024**

1. A fast-food restaurant operates both a drivethrough facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-through and walk-in facilities are in use. The joint density function of these random variables is

$$f(x,y) = \begin{cases} k(x+2y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(13 points)}$$

   (a) What is the value of k that makes the above a valid density function? (3 points)

   (b) What is the conditional probability of X given Y? (3 points)

   (c) Are X and Y independent? Justify your answer. (4 points)

   (d) What is the probability that the drive-through facility is busy less than $\frac{1}{4}$ of the time? (3 points)

2. Consider a random sample of $X_1, \ldots, X_n$ from a uniform distribution $U(0, \theta)$ with unknown parameter θ, where $\theta > 0$. (7 points)

   (a) Compute $E(X_1)$ and $Var(X_1)$. (2 points)

   (b) Explain that the maximum likelihood estimator of θ is $X_{(n)} \equiv \max\{X_1, \ldots, X_n\}$, which is the largest order statistic of the sample. (2 points)

   (c) The probability density function of $X_{(n)}$ in (b) is given as

$$f_{X_{(n)}}(x) = n\frac{x^{n-1}}{\theta^n}.$$

   Show that $E(X_{(n)}) = \dfrac{n}{n+1}\theta$. Next, based on the fact, obtain an unbiased estimator of θ. (3 point)

3. Suppose you have two normal populations. Their means are denoted as μ_1 and μ_2. Both populations have equal variance σ^2. All μ_1, μ_2, and σ^2 are unknown. A random sample of size n_1, $X_{11}, X_{12}, \ldots, X_{1n_1}$ is drawn from the first population. A random sample of size n_2, $X_{21}, X_{22}, \ldots X_{2n_2}$ is drawn from the second population. (10 points)

   (a) Write down a pooled estimator of σ^2, denoted as S_p^2. (1 points)

   (b) Which distribution does $\dfrac{(n_1+n_2-2)S_p^2}{\sigma^2}$ follow? What about the degrees of freedom? (1 point)

   (c) Derive a test statistic that can be used to test $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$. (3 points)

   (d) Which distribution does the test statistic in (c) follow? What about the degrees of freedom? (1 point)

   (e) At the 0.05 level of significance, when do we conclude to reject H_0? (2 points)

   (f) Construct a 90% confidence interval for $\mu_1 - \mu_2$. (2 points)