Analysis I - 2024 Spring Graduation Exam

1. (10) Let $\{a_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}$ that satisfies

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Show that $\lim_{n\to\infty} a_n$ exists.

2. (10) Suppose that $f : [0,1] \to [0,1]$ is a differentiable function such that $|f'(x)| \leq C$ for all $x \in [0,1]$ for some $C \in (0,1)$. Let a_0 be in $(0,1)$ and set $a_{n+1} = f(a_n)$.

(a) Show that the equation $f(x) = x$ has at most one solution.

(b) Prove that the sequence a_n tends to the unique root of $f(x) = x$ in $[0,1]$.

3. (10) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq |x-y|^{3/2}.$$

Show that f is a constant function.