1. 2023 Calculus graduation Exam

Problem 1. 10 points

The Stokes' Theorem is: The circulation of a vector field $\mathbf{F} = (P, Q, R)$ around the boundary ∂S of an oriental surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S, in other words,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathrm{t} = \int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \quad -(1)$$

Let S be the hemisphere defined by $x^2 + y^2 + z^2 = 16, z \geq 0$ and $C = \partial S$ its bounding circle defined by $x^2 + y^2 = 16, z = 0$ and the vector field: $\mathbf{F} = (y, -x, 0)$. For this case, show that the equation in (1) holds.

Problem 2. 10 points

Find the maximum and minimum values of the function $f(x, y) = 2x + 3y + 1$ on the circle $x^2 + y^2 = 5$. (Hint: Use the method of the Lagrange multiplier).

Problem 3. 10 points

Let f be a twice continuously differentiable function on an interval I containing a and b. Show that

$$\left|\frac{1}{2}(f(a) + f(b)) - f((a+b)/2)\right| \leq \frac{1}{8} M (b-a)^2$$

where $M = \max_{x \in I} |f''(x)|$.

(Hint: Use the Taylor expansion of the function at two points a, b.