Applied Linear Algebra

Choose three out of the four problems. (3×10pts)

1. Let A be a Hermitian matrix

   (a) Show that eigenvalues of A are real.

   (b) Show that two eigenvectors of A with distinct eigenvalues are orthogonal to each others.

2. Let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be a self-adjoint operator on $\mathbf{R}^2$ with the Euclidean inner product, $\langle v, w \rangle = v \cdot w$ for every pair $v, w \in \mathbf{R}^2$: If we use the column form $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of vectors v with respect to the standard basis of $\mathbf{R}^2$, we have $\langle v, w \rangle = v^T w$ and $S(v) = Av$, where A is a real symmetric 2×2 matrix representing S.

   (a) Consider the following real valued function k on the space of all unit vectors on $\mathbf{R}^2$ given by

   $$k(u) = \langle S(u), u \rangle, \quad \text{where } u \in \mathbf{R}^2 \text{ and } \langle u, u \rangle = 1.$$

   Show that the maximum and the minimum of the function k are the eigenvalues of unit eigenvectors of S. (hint. You may use the matrix notations and the method of Lagrangian multiplier).

   (b) Show that, for every $v, w \in \mathbf{R}^2$, we have

   $$\langle S(v), S(w) \rangle - \operatorname{tr}(S) \, \langle S(v), w \rangle + \det(S) \, \langle v, w \rangle = 0.$$

   (hint. You may use matrix notations for the left-hand-side of the above.)