***** 문제, 뒷면에도 있음 *****

**Qualifying Examination -Analysis - 2023 July**

**1.** (*10 points*) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there is $r \in (0, 1)$ such that $|f'(x)| \leq r$ for all $x$. Let $a_1 \in \mathbb{R}$ and define $a_{n+1} = f(a_n)$. Show that the sequence $a_n$ converges.

**2.** (*10 points*) For real sequences $\{a_n\}$ and $\{b_n\}$, assume $\sum a_n$ converges, and $\{b_n\}$ is monotone and bounded. Show that $\sum a_n b_n$ also converges.

**3.** (*10 points*) Is closed and bounded set compact always? Prove or provide a counter example.

**4.** (*15 points*) Consider a sequence of real-valued functions

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(1) (10pts) Show that $f_n \to f$ uniformly for some $f$.
(2) (5pts) Does $f'(x) = \lim_{n\to\infty} f_n'(x)$ hold for all $x \in \mathbb{R}$?

**5.** (*15 points*) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} dx, \; -\infty < t < \infty.$$

**6.** (*10 points*) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $u(x, y) = x^3 - 3xy^2$. Show that $u$ is harmonic and find $v : \mathbb{R}^2 \to \mathbb{R}$ such that the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x + iy) = u(x, y) + iv(x, y)$ is analytic.

**7.** (*10 points*) Let the points $a, b,$ and $c$ lie on the unit circle of the complex plane and satisfy $a + b + c = 0$. Prove that $a, b,$ and $c$ form the vertices of an equilateral triangle.

**8.** (*10 points*) [Second-order Runge-Kutta method] For a continuous real function $f : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$, we have an ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad \text{with } x(0) = x_0.$$

Assume that $\frac{\partial}{\partial t} f(t, x)$ and $\frac{\partial}{\partial x} f(t, x)$ exists for any $t$ and $x$. Let $0 < t_1 < t_2 < \cdots < t_N$ with $t_{i+1} - t_i = h$ for a sufficiently small $h$. Define for

$i = 0, 1, 2, \ldots, N-1$,

$$x_{i+1} = x_i + \frac{h}{2}(K_1 + K_2), \text{where } K_1 = f(t_i, x_i) \text{ and } K_2 = f(t+h, x_i + hK_1).$$

Show that $|x(t_i) - x_i| \le C_i h^3$ with some positive constant $C_i$ for each $i = 0, \ldots, N$.

(Hint: use the second order Taylor expansion for $x(t)$ and first order Taylor expansion of $f(t, x)$.)

**9.** (*5 points*) Let $y = f(x)$ satisfy the following ordinary differential equation:

$$(x+2)\sin y + x\cos y \frac{dy}{dx} = 0.$$

Find the (implicit) general solution of this differential equation by using an integrating factor $\mu(x)$ such that $\mu(x)(x+2)\sin y + \mu(x)x\cos y\frac{dy}{dx} = 0$ is an exact differential equation in $\mathbb{R}^2$. (i.e. $\frac{\partial}{\partial y}\left[\mu(x)(x+2)\sin y\right] = \frac{\partial}{\partial x}\left[\mu(x)x\cos y\right]$ for any $x$ and $y$.)

**10.** (*5 points*) Find $r_1$ and $r_2$ such that any solution $y(x)$ to $2x^2y'' + xy' - y = 0$ can be written as $y(x) = a_1 x^{r_1} + a_2 x^{r_2}$ with some constants $a_1$ and $a_2$.