QE(ALGEBRA), 2023 JULY

1 (14 pt) Show that every group of order 105 is not simple.

2 (14 pt) Construct a non-cyclic infinite group G in which all proper subgroups are cyclic.

3 (14 pt) Show that an integral domain cannot contain two subrings isomorphic to $\mathbb{Z}/p$ and $\mathbb{Z}/q$ for some two distinct primes p and q.

4 (14 pt) Suppose that every ideal of an integral domain R is finitely generated. Prove that every ideal of $R[x]$ is finitely generated.

5 (14 pt) Suppose that E is a Galois extension over a field K, and K is Galois over a field F. Is E Galois over F? Prove or disprove it.

6 (14 pt) Let G be the Galois group of $x^8 - 4x^4 - 1$ over $\mathbb{Q}$ and denote by

$$H_0 = \{e\} \ < \ \cdots \ < \ H_n = G$$

a composition series of G. Find n and H_{i+1}/H_i for each $0 \leq i \leq n-1$.

7 A group G is called Galois if there is a Galois extension K over a field F such that G is isomorphic to $\mathrm{Gal}(K/F)$.

(a) (7 pt) Show that the symmetric group of degree n, i.e. S_n, is Galois.

(b) (7 pt) Show that every finite group is Galois.