Qualifying Examination -Analysis - 2023 January

1. (*10 points*) (a) Show that

$$\lim_{p\to\infty} \|f\|_{L^p[0,1]} = \sup_{x\in[0,1]} |f(x)|$$

for any continuous function f on $[0,1]$.
(b) What if we consider $[0,\infty)$ instead of $[0,1]$?

2. (*10 points*) Let $f(x)$ be a continuous function on $[0,\infty)$ such that $\lim_{x\to\infty} f(x) = L$ for some $L \in \mathbb{R}$. Prove

$$\lim_{n\to\infty} \int_0^1 f(nx)dx = L.$$

3. (*10 points*) Prove that

$$\lim_{n\to\infty} \frac{1}{n}(n!)^{\frac{1}{n}} = e^{-1}.$$

4. (*15 points*) Consider $f_n(x) = \sin(nx)$ for $-\pi \le x \le \pi$ and $n = 1, 2, 3, \cdots$.
(a) (5 pts) Show that $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^2[-\pi,\pi]$ where $L^2[-\pi,\pi]$ is square Lebesgue integrable function space.

(b) (5 pts) Show that $\{f_n\}_{n=1}^{\infty}$ is closed in $L^2[-\pi,\pi]$.

(c) (5 pts) Is $\{f_n\}_{n=1}^{\infty}$ compact in $L^2[-\pi,\pi]$? Prove or disprove.

5. (*15 points*) Evaluate the integral

$$\int_0^\infty \frac{1-\cos ax}{x^2}dx$$

for $a \in \mathbb{R}$.

6. (*10 points*) Let f be an entire function such that $\mathrm{Re}(f(z)) \ge -2$ for all $z \in \mathbb{C}$. Show that f is constant.

7. (*10 points*) Let f be a nonconstant entire function whose values on the real axis are real and nonnegative. Prove that all real zeros of f have even order.

8. (*5 points*) Find the general solution of $y'' - 3y' + 2y = te^{3t}$.

9. (*5 points*) Find the general solution of

$$(2xt + e^x)\frac{dx}{dt} + (x^2 + \sin t) = 0.$$

The general solution can be implicitly expressed.

10. (*10 points*) Let $x(t)$ be the solution to the initial problem

$$x' = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 7/5 \\ 5/2 \end{pmatrix}.$$

Then show that there exists a constant $C > 0$ such that

$$\|x(t) - x^*\|_1 \le Ce^{-t},$$

where $x^* = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.