Algebra Qualifying Exam

January 2023

Do SIX problems out of the following seven problems.

(1) [20 points] Find all possible ring homomorphisms $\phi : \mathbb{Z}_{11} \to \mathbb{Z}_{33}$. Justify your anwer.

(2) [20 points]
  (a) Find a unique factorization domain (UFD) which is not a principal ideal domain (PID).
  (b) Find an integral domain which is not a UFD.

(3) [20 points] Let K be a finite field extension of degree n over $\mathbb{Q}$. Choose an element α in $K \setminus \mathbb{Q}$ and suppose that the minimal polynomial for α over $\mathbb{Q}$ is

$$m_{\alpha,\mathbb{Q}}(t) = t^d + a_1 t^{d-1} + a_2 t^{d-2} + \cdots + a_{d-1} t + a_d.$$

We regard K as a $\mathbb{Q}$-vector space. We then define a $\mathbb{Q}$-linear operator

$$T_\alpha : K \to K$$

by $T_\alpha(v) = \alpha v$.
  (a) [6 points] What is the characteristic polynomial of T_α?
  (b) [6 points] Does T_α have a Jordan canonical form? Justify your answer.
  (c) [8 points] Describe the rational canonical form of T_α.

(4) [20 points] Denote by $\mathbb{C}(z)$ the field of rational functions in one variable z. Prove that the group of automorphisms of $\mathbb{C}(z)$ that leave the field $\mathbb{C}$ of constant rational functions fixed is isomorphic to the quotient group of the general linear group $\mathrm{GL}(2, \mathbb{C})$ by its center. Note that the center of $\mathrm{GL}(2, \mathbb{C})$ is $\{\lambda I \mid \lambda \in \mathbb{C}^*\}$.

(5) [20 points]
  (a) Construct a finite field K of order 7^3.
  (b) Is K Galois over $\mathbb{Z}_7$? (Here, $\mathbb{Z}_7$ is considered as a subfield of K of order 7.)

(6) [20 points] Let $f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x]$.
  (a) Show f is irreducible over $\mathbb{Q}$.
  (b) Find the Galois group of the splitting field of f.

(7) [20 points] Let K be the splitting field of $x^6 - 7^{\frac{1}{5}}$ over $\mathbb{Q}(7^{\frac{1}{5}}, e^{\frac{2\pi i}{6}})$. Is $\mathrm{Gal}(K/\mathbb{Q}(7^{\frac{1}{5}}, e^{\frac{2\pi i}{6}}))$ abelian? Justify your answer.

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