Graduation Examination (2023-1)
Analysis I

1. (*10 points*) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove that there exists a positive number B such that $|f(x)| \leq 1 + B|x|$, for all x.

2. (*10 points*) Let $f(x) = x\log(1+\frac{1}{x})$, for $x > 0$. Show that f is strictly increasing and compute $\lim f(x)$ as $x \to 0$ and $x \to \infty$.

3. (*10 points*) Suppose $x_1,\ x_2,\ x_3,\ \ldots$ is a sequence of non-negative real numbers satisfying

$$x_{n+1} \leq x_n + \frac{1}{n^2}$$

for all $n \geq 1$. Prove that $\lim_{n\to\infty} x_n$ exists.