Probability and Statistics Graduation Exam

1. For two continuous random variables X and Y, it is known that the joint probability distribution is

$$f(x, y) = \begin{cases} 6x^2y & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $P(X \geq 1/2)$. (3 points)

   (b) Are X and Y independent? Justify your answer. (4 points)

   (c) Compute $Cov(X, Y)$, which means the covariance between X and Y (3 points)

2. Let X and Y follow Poisson distributions with the rate $\lambda > 0$ and $\mu > 0$, respectively. Assuming X and Y are independent, show that $X + Y$ follows a Poisson distribution with the rate $\lambda + \mu$. (10 points)

3. You sampled 125 batteries and measured the lifespan of the batteries from a car battery manufacture. It is found that the sample mean is 3 years. Assuming the sample size 125 is large enough for using Central Limit Theorem, find a 95% confidence interval for the mean of the lifespan of the batteries produced from the manufacture. (For Z that follows the standard normal distribution, we have $P(Z \geq 1.96) = 0.025$). (10 points)