Graduation Examination (Fall 2022)
Analysis I

1. (*10 points*) Let X be a metric space and $\{x_n\}$ be a convergent sequence in X with limit x_0. Prove that the set $C = \{x_0, x_1, x_2, x_3, ...\}$ is compact.

2. (*10 points*) Let the real-valued function $f : [0,1] \to \mathbb{R}$ satisfy the following two properties:

- (Intermediate Value Property) If $[a,b] \subset [0,1]$, then $f([a,b])$ contains the interval with endpoints $f(a)$ and $f(b)$.
- For each $c \in \mathbb{R}$, the set $f^{-1}(c) := \{x \in [0,1] : f(x) = c\}$ is closed.

Prove that f is continuous.

3. (*10 points*) Define a sequence of real numbers $\{x_n\}$ by

$$x_0 = 1, \quad x_{n+1} = \frac{1}{2 + x_n} \quad \text{for} \quad n \geq 0.$$

Show that $\{x_n\}$ converges. Evaluate its limit.