Qualifying Examination - Analysis - July 2022.

1. (*10 points*) f is a real-valued continuous function on $[0, 1]$ such that

$$\int_0^1 f(x^{1/n})dx = 0, \quad n = 1, 2, 3, \cdots .$$

Prove that $f \equiv 0$.

2. (*10 points*) Prove that

$$\lim_{n\to\infty} \frac{1}{n}(n!)^{\frac{1}{n}} = e^{-1}.$$

by considering integration of $\log x$.

3. (*15 points*) Consider the Riemann-Stieltjes integral with $\alpha(x) = \sum_{n=1}^{\infty} I(x - n)$ where

$$I(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

Real function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 1 & x = y, \\ -1 & y = x + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Check whether the following is true or not.

$$\int_{y>-\frac{1}{2}} \int_{x>-\frac{1}{2}} f(x, y)d\alpha(x)d\alpha(y) = \int_{x>-\frac{1}{2}} \int_{y>-\frac{1}{2}} f(x, y)d\alpha(y)d\alpha(x)$$

(We define $\int\int f(x, y)d\alpha(x)d\alpha(y) := \int[\int f(x, y)d\alpha(x)]d\alpha(y)$)

4. (*15 points*) Consider the following integral equation,

$$f(x) = \lambda \int_a^x K(x, y)f(y)dy + \varphi(x), \quad \lambda \in \mathbb{R},$$

where $K(x, y)$ is a real continuous function on $[a, b] \times [a, b]$ and φ is also a given real continuous function on $[a, b]$.

(a) (5 points) Let define $\Phi(f)(x)$ for $f \in \mathcal{C}([a, b], \mathbb{R})$ (continuous real function defined in $[a, b]$) as

$$\Phi(f)(x) = \lambda \int_a^x K(x, y)f(y)dy + \varphi(x).$$

Show that $\Phi(f) \in \mathcal{C}([a, b], \mathbb{R})$.

(b) (10 points) Show that the integral equation $\Phi(f) = f$ has a unique solution $f \in \mathcal{C}([a,b],\mathbb{R})$ for any λ.

5. (*10 points*) Let a and b be complex numbers whose real parts are negative or 0. Prove the inequality

$$|e^a - e^b| \leq |a - b|.$$

6. (*10 points*) For each $k > 0$, let X_k be the set of analytic functions $f(z)$ on the open unit disc $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \left((1-|z|)^k |f(z)|\right)$$

is finite. Show that $f \in X_k$ if and only if $f' \in X_{k+1}$.

7. (*10 points*) Does there exist a function f, analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that

$$|f(z)| \geq \frac{1}{\sqrt{|z|}},$$

for all nonzero z?

8. (*10 points*) Find at least two solutions of the initial value problem

$$y' = \sqrt{y}, \quad y(0) = 0.$$

Explain why there does not exist a unique solution.

9. (*10 points*) Find the solution of the following initial value problem

$$\boldsymbol{x}'(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$