Algebra Qualifying Exam

July 2022

Your score will be the sum of credits below multiplied by 10/9. Total score is 100.

1. (15 pts) Let A be an n by n matrix. If $A^m = I$ for some $m \in \mathbb{N}$, is A is diagonalizable? Prove or disprove it.

2. (15 pts) List, by giving generators for them, all prime ideals of $\mathbb{Z}[x]/\langle 12, x^2 + x + 1\rangle$. Caution that you should not include 0 as a generator of your ideal. You must justify your answer.

3. (15 pts) Let $\mathbb{F}_q$ be the finite field of q elements.
   (a) (5 pts) For an arbitrary prime integer p, prove that the number of monic irreducible polynomials of degree p in $\mathbb{F}_q[x]$ is $\frac{q^p - q}{p}$.
   Hint: Consider the splitting field of such polynomial.
   (b) (5 pts) Find the number of conjugacy classes in the group $\mathrm{GL}_5(\mathbb{F}_q)$ which does not have eigenvalues in $\mathbb{F}_q$. You must justify your answer.
   (c) (5 pts) Find the number of conjugacy classes in the group $\mathrm{GL}_5(\mathbb{F}_q)$ which does not have eigenvalues in $\mathbb{F}_q$ but have eigenvalues in the quadratic field extension of $\mathbb{F}_q$. You must justify your answer.

4. (15 pts)
   (a) (5 pts) Show $\mathbb{Z}[x]/ < x^2 + 1 >$ is an Euclidean domain.
   (b) (5 pts) Using the fact that $\mathbb{Z}[x]/ < x^2 + 1 >$ is a Euclidean domain, describe all the integer points of $x^2 + y^2 = z^2$.
   (c) (5 pts) Is $\mathbb{Z}[x]/ < x^2 + 5 >$ a Euclidean domain? Prove or disprove it.

5. (15 pts) Determine if the following statement is true. Justify your answer.: The splitting field in $\overline{\mathbb{Q}}$ of $x^{70} - 1 \in \mathbb{Q}[x]$ should be the same as the splitting fields in $\overline{\mathbb{Q}}$ of $(x^7 - 1)(x^{10} - 1)$ over $\mathbb{Q}$.

6. (15 pts) Determine if the following statement is true. Justify your answer.: Let $\alpha = \sqrt{3 - \sqrt{3}}$. Then $\mathbb{Q}(\alpha)$ should be a normal extension over $\mathbb{Q}$ with a cyclic Galois group $Gal(\mathbb{Q}(\alpha)/\mathbb{Q})$.
