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Abstract

Let R be an integrally closed integral domain, $\{X_\alpha$ a set of indeterminates over R, and T a multiplicatively closed subset of $R[\{X_\alpha\}]$. We prove the equivalence of the following statements: (1) Every prime ideal of $R[\{X_\alpha$ $]_T$ is extended from R. (2) Every ideal of $R[\{$ $X_\alpha]_T$ is extended from R. (3) Every principal ideal of $R[\{X_\alpha\}]_T$ is extended from R. (4) There exists a Prüfer v-multiplication overring A of R such that $R[\{X_\alpha\}]_T = A^v$, where A^v is the Kronecker function ring of A with respect to the v-operation. The case when R is not integrally closed is also taken care of. Similar statements for rings with zero divisors are considered and their equivalence is established.

I Introduction

In the literature quotient rings of $R[X]$ have been considered [?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. The most frequent ones are $R\langle X\rangle$, $R(X)$, and $R[X]_{N_v}$ (for definitions see Section ??). Characterizations of R are given when the prime ideals of $R(X)$, $R[X]_{N_v}$ are extended from R 2,HiHu77,4,6,7,8,9,Mat83b,Mat83a,Zaf84.

Let R be an integrally closed integral domain with quotient field K.

Arnold [?, Theorem 4] showed that R is a Prüfer domain if and only if the prime ideals of $R(X)$ are extended from R. In [?, Theorem 3.1], it was shown that if R is a GCD-domain, an integrally closed coherent domain, or a Krull domain, then $R[X]_{N_v}$ is a Bezout domain. A GCD-domain, an integrally closed coherent domain, and a Krull domain belong to the same class of rings, namely Prüfer v-multiplication domains (PVMD's). In [?, ?] it was proven that if R is a PVMD, then $R[X]_{N_v}$ is a Bezout domain, and the converse was established too. In [?, Theorem 3.1], Kang proved that the prime ideals of $R[X]_{N_v}$ are extended from R if and only if all ideals of $R[X]_{N_v}$ are extended from R. For an arbitrary multiplicative set T of $R[X]$, Huckaba and Papick's result [?, Lemma 3.0] states that $R[X]_T$ is a Bezout domain if the prime ideals of $R[X]_T$ are extended from R (we write Spec (R[X]$_T$) $=\mathrm{Spec}(R)^e$). In Theorem ??, we give a complete characterization of the domain R and the multiplicative set T such that $\mathrm{Spec}(R[X]_T)=\mathrm{Spec}(R)^e$: Spec (R[X]$_T$) $=\mathrm{Spec}(R)^e \Leftrightarrow$ every ideal of $R[X]_T$ is extended from $R \Leftrightarrow$ there exists a PVMD overring A of R such that $R[X]_T=A[X]_{N_v(A)} \Leftrightarrow R[X]_T$ is a Bezout domain and $N_v(D)$ is the saturation of T in $D[X]$, where D=R[X]$_T \cap K \Leftrightarrow R[X]_T$ is a Prüfer domain and N$_v(D)$ is the saturation of T in $D[X] \Leftrightarrow$ fR[X]$_T = A_f[X]_T$ for all $f \in R[X]$, where A_f is the ideal of R $generatedbythecoefficientsof$f$.Inparticular, theprimeidealsof$ $R[X]_T$ are extended from R if and only if $R[X]_T$ is the Kronecker function ring of a PVMD with respect to the v-operation.

For an integral domain which is not necessarily integrally closed, a combination of [?, Lemma 4.2] and [?, Theorem 3(a)] implies that $\mathrm{Spec}(R(X))=\mathrm{Spec}(R)^e$ if and only if the integral closure R' of R is a Prüfer domain. In fact (Proposition ??), for an integral extension $R \subseteq D$, $\mathrm{Spec}(R[X]_T)=\mathrm{Spec}(R)^e$ if and only if (D[X]$_T$) $=\mathrm{Spec}(D)^e$. Using this, we obtain another characterization (Theorem ??): Let $K \leq L$ be a field extension and R' be the integral closure of R in L. Then Spec (R[X]$_T$) $=\mathrm{Spec}(R)^e$ if and only if $D=R'[X]_T \cap K$ is a PVMD and $N_v(D)$ is the saturation of T in $D[X]$. An interesting result (Corollary ??) is that for a Noetherian domain R, N$_v(R)'=N_v(R')$ if and only if t-dim (R)≤ 1.

Let R be a ring with zero divisors. In this case, we assume that the multiplicative subset T of $R[X]$ does not contain a zero divisor of $R[X]$. Some results on integral domains were generalized to additive regular v-rings with property (A) (for the results and relevant definitions, see [?, Theorem 3.6]). Also the following can be found in [?, Theorems 21.2, 22.5]: (1) If R is an integrally closed Marot ring with property

(A)
and $\operatorname{Spec}(R[X]_T) = \operatorname{Spec}(R)^e$, then $R[X]_T$ is a
Bezout ring. (2) If R is an integrally closed additively regular v-ring,
then $\operatorname{Spec}(R[X]_{N_v}) = \operatorname{Spec}(R)^e \Leftrightarrow R$
is a PVMR $\Leftrightarrow R[X]_{N_v}$ is a PVMR $\Leftrightarrow$
R[X]$_{N_v}$ is a Prüfer ring $\Leftrightarrow R[X]_{N_v}$ is a
Bezout ring. We extend the result (2) to a larger class of rings, namely to
Marot rings with property
(A)
. For a ring which is not necessarily integrally closed, similar results to
the domain case are obtained.

II Preliminaries

Throughout this paper, R will be a commutative ring
with identity and possibly with zero divisors. We denote by K the total
quotient ring of R. For an R-submodule M of K, M^{-1} shall stand
for the R-submodule $\{x \mid x \in K,\ xM \subseteq R\}$ and
M$_v$ is defined to be $(M^{-1})^{-1}$. The submodule M_t is defined as
the submodule $\bigcup\{(M')_v \mid M'$ is a finitely
generated submodule of $M\}$. An ideal I of R is a v-ideal (resp. t
-ideal) if $I_v = I$ (resp. $I_t = I$). An ideal I is said to be t
-invertible if $(II^{-1})_t = R$. The set of zero divisors of R is denoted
by $\mathbf{Z}(R)$. A non-zero-divisor of R is called a regular element
and an ideal that contains a regular element is called a regular ideal. A
ring is called a Bezout ring, Prüfer ring, or Prüfer v
-multiplication ring (PVMR) if every finitely generated regular ideal is
principal, invertible, or t-invertible respectively. Let $\{X_\alpha\}$
be an arbitrary set of variables. For brevity of notation, we will denote
the set $\{X_\alpha\}$ by X. Let T be a multiplicative subset of
R[X] $such that$ T$\cap\mathbf{Z}(R[X]) = \varnothing$. We use the notation
T$'$ or $\bar{T}$ for the saturation of T. For an element $f \in$
R[X],A$_f$ is the ideal of R generated by the coefficients of f. The
multiplicative subset $\{f \in R[X] \mid A_f = R\}$ (resp. $\{f \in R[X] \mid$
A$_f$ is a regular ideal and $(A_f)_v = R\}$) will be denoted by $N(R)$ (resp. $N_v(R)$).

From the Dedekind--Mertens lemma [?, Corollary 28.3], it easily follows
that both N and N_v are disjoint from $\mathbf{Z}(R[X])$. We use the notation $R(X)$ and $R\{X\}$ for the rings $R[X]_N$ and $R[X]_{N_v}$ respectively. For undefined terms and terminology the readers are referred
to [?, ?, ?]. We will consider the following three statements.

(PEI) Every regular prime ideal of $R[X]_T$ is extended from an ideal of R.

(PEP) Every regular prime ideal of $R[X]_T$ is extended from a prime ideal of R.

(IEI) Every regular ideal of $R[X]_T$ is extended from an ideal of R.

In case that PEI (resp. PEP, IEI) is satisfied, we say that (R,T) is a
PEI-pair (resp. PEP-pair, IEI-pair). Also we use the notation (R[X]$_T$) $= \mathrm{Spec}(R)^e$ when (R,T) is a PEP-pair.
In case R is a domain and $(R,N_v(R))$ is a PEP-pair, R is also called a UMT-domain [?, Theorem 3.1].

In [?], Arnold proved that if R is an integrally closed domain and

(R,N(R))$isaPEP-pair, then$R(X)$isaPrüferdomain.The$
u-or-u^{-1} lemma used in his proof can be generalized to rings with zero divisors and using this lemma it was shown that for an integrally closed Marot ring with property
(A)
, if (R,T) is a PEP-pair, then $R[X]_T$ is a Bezout ring in the single variable case [?, Theorem 21.2]. Through a different approach we will
extend this result to an arbitrary set of variables. In the mean time, it
will be shown that for an integrally closed domain, all the three statements PEI, PEP, and IEI are equivalent; we will give a complete characterization of such domains and multiplicative sets involved (Theorem
??). In Section ?? we extend these results to rings with zero divisors. To achieve the goal, we need to extend necessary results on
integral domains to rings with zero divisors.

Let R be a ring with zero divisors. Conventionally, a $*$-operation is
defined on the fractional (regular) ideals of R [?, §32]. However,
in the obvious way, it can be extended to arbitrary ideals of R. Let
F(R)$bethesetof$R$-submodulesof$K$.Thenafunction$*
:
F(R)$\rightarrow \mathcal{F}(R)$ is a star operation if the
function $*$ satisfies the following properties for $A, B \in \mathcal{F}(R)$
and $a \in R$: (1) $aA_* \subseteq (aA)_*$; and if a is regular, then
a$A_* = (aA)_*$. (2) $A \subseteq A_*$, and $A \subseteq B$ implies
A$_* \subseteq B_*$. (3) $(A_*)_* = A_*$. (4)
(A+B)$_* = (A_* + B_*)_*$. (5) $(AB)_* = (AB_*)_* = (A_*B_*)_*$.
Properties (4) and (5) can derived from (1), (2), and (3). For details see
[?, Lemma 20.1]. The v-operation is a $*$-operation
[?, Theorem 20.9] and in a Marot ring, it is the biggest $*$-operation
in the sense that $(A_*)_v = A_v$ for any regular ideal A of R
[?, Proposition 32(iv)].
A typical example of a $*$-operation is one induced by overrings: suppose a
ring R is of the form $R = \bigcap R_\alpha$, where R_α is a
collection of overrings of R. The function $*$ defined by $I_* = \bigcap$
(IR$_\alpha$) for an ideal I of R is a $*$-operation.
Recall that a Marot ring is a ring whose every regular ideal is generated by
regular elements. The following result is well-known for integrally closed
integral domains.

Proposition 2.1 *Let R be an integrally closed Marot ring. Then
$(A_f A_g)_v = (A_{fg})_v$ for any $f, g \in R[X]$ such that A_f and
A_g are regular ideals.*

Proof. By [?, Proposition 14], R is the
intersection of all the valuation overrings of R, say $R = \bigcap V_\alpha$
$.Let*bethestaroperationon$R$inducedbythevaluationoverrings$
V$_\alpha$. Now $(A_f A_g)_* = \bigcap (A_f A_g) V_\alpha = \bigcap$
(A$_f V_\alpha)(A_g V_\alpha) = \bigcap (A_{fg}) V_\alpha$
=(A$_{fg})_*$. The third equality follows from [?, Proposition 17]
and the Dedekind--Mertens lemma [?, Corollary 28.3]. From
[?, Proposition 32(iv)], we deduce that $(A_f A_g)_v = (A_{fg})_v$
. □

Proposition 2.2 (cf. [?, Corollary 2.3, Proposition 2.8] and [?, Lemma 1].)

Let R be a ring with the total quotient ring K and $\mathrm{Maxt}(R)$ the set of all regular maximal t-ideals of R. Then:

(1) *For a regular ideal I of R,* $I[X]^{-1} = I^{-1}[X], I[X]_v = I_v[X]$, *and* $I[X]_t = I_t[X]$.

(2) *For a multiplicative subset S of R such that $S \cap \mathbf{Z}(R) = \varnothing$, and an ideal I of R, $(I_S)_t = ((I_t)_S)_t$. Especially for $N_v(R)$ in $R[X]$ and a regular ideal I of R, we have $(I[X]_{N_v(R)})^{-1} = I^{-1}[X]_{N_v(R)}$, $(I[X]_{N_v(R)})_v = I_v[X]_{N_v(R)}$, and $(I[X]_{N_v(R)})_t = I_t[X]_{N_v(R)}$. Furthermore if I is a t-ideal of R, then $I[X]_{N_v(R)} \cap K = I$.*

(3) $R = \bigcap_{M\in \mathrm{Maxt}(R)} R_{(M)}$, *where $R_{(M)}$ is the ring R localized at the set of regular elements of R contained in $R \setminus M$.*

Proof. (??) and (??) are straightforward. Although (??) is stated for a Marot ring R in [?, Lemma 1], the proof given there is valid for an arbitrary ring. □

A ring R satisfies the property (A) if $f \in R[X]$ is regular $\Leftrightarrow A_f$ is a regular ideal of R.

Proposition 2.3 *Let R be an integrally closed Marot ring with property (A) and J a regular t-ideal of $R[X]$. Then J is extended from R if and only if $J \cap R$ contains a regular element of R.*

Proof. ($\Rightarrow$) For an ideal I of R, $J = I[X]$. Choose a regular element $f \in J$. By the property (A), A_f is a regular ideal. So I ($\supseteq A_f$) is a regular ideal.

($\Leftarrow$) Let $a \in J \cap R$ be a regular element. Let $f \in J$ be a regular element. Since $a(a,f)^{-1}$ is a regular ideal of $R[X]$ and $R[X]$ is a Marot ring, $a(a,f)^{-1}$ is generated by regular elements. Let $g \in a(a,f)^{-1}$ be a regular element. Now $A_{fg} \subseteq aR$. From [?, Lemma 3.3(5)], it follows that $(A_{fg})_v \subseteq (a)_v = (a)$. By Proposition ??, $(A_f A_g)_v = (A_{fg})_v$. Thus we get

$A_f A_g \subseteq (a)$, i.e., $g/a \in A_f^{-1}[X]$ which is $(A_f[X])^{-1}$ by Proposition ??(??). Finally $(a,f)^{-1} \subseteq (A_f[X])^{-1}$ and so $(A_f[X])_v \subseteq (a,f)_v \subseteq J$. Let $h \in J$. Then there exist regular elements $f_1, \ldots, f_n \in J$ such that $h \in (f_1, \ldots, f_n)$. Clearly $A_h \subseteq A_{f_1} + \cdots + A_{f_n} \subseteq J$ and therefore $J=(J \cap R)[X]$. □

In Lemmas ??, ??, and ??, we assume that R is an integrally closed Marot ring with the property (A).

Lemma 2.4 *Let B be the set of all regular t-ideals of $R[X]_T$ which are* not *extended from R. If $B \neq \varnothing$, then B has a maximal element and every maximal element is a prime t-ideal.*

Proof. Let J be a regular t-ideal of $R[X]_T$. From Proposition ??(??), we deduce that $J \cap R[X]$ is a regular t-ideal. By Proposition Pro3, J is extended from R if and only if $J \cap R$ contains a regular element of R. Suppose $B \neq \varnothing$. By Zorn's lemma B has a maximal element, say J. Let $Q = J \cap R[X]$. It suffices to show that Q is a prime ideal of $R[X]$. Obviously Q is a t-ideal by Proposition Pro2. Suppose $f, g \in R[X] \setminus Q$ and $fg \in Q$. Then there exist regular elements $a \in (Q,f)_t \cap R$ and $b \in (Q,g)_t \cap R$. Now the regular element $ab \in ((Q,f)_t(Q,g)_t)_t \subseteq Q$, which contradicts that $J = Q_T$ is not extended from R (Proposition ??). So $fg \notin Q$. *Thus* Q *is a prime ideal.* □

We show that PEP is equivalent to IEI.

Lemma 2.5 *Every regular prime ideal $R[X]_T$ is extended from a prime ideal of R if and only if $fR[X]_T = A_f[X]_T$ for any regular element f of $R[X]$.*

Proof. ($\Rightarrow$) Let Q be a regular prime ideal of $R[X]$ such that $Q \cap T = \varnothing$. Then, by the assumption, Q is of the form $Q = P[X]$ where P is a prime ideal of R.
By Lemma ??, every regular principal ideal of $R[X]_T$ is extended from an ideal of R. Let f be a regular element of $R[X]$, so of

R[X]$_T$. Then for some ideal I of R, $fR[X]_T = I[X]_T$
$\Rightarrow fR[X]_Q = I[X]_Q \Rightarrow fR[X]_{P[X]} = I[X]_{P[X]}$
$\Rightarrow fR_P(X) = IR_P(X) \Rightarrow fR_P(X) = A_f R_P(X)$
by [?]. $\Rightarrow fR[X]_Q = A_f R[X]_Q \Rightarrow$
J:=[fD:$_D A_f D$] is a regular ideal of $D = R[X]_T$, which is not contained in any prime ideal of D (note that any prime ideal containing J
is regular since J is regular). $\Rightarrow J = D$. So $fD = A_f D$,
i.e., $fR[X]_T = A_f[X]_T$.
($\Leftarrow$) Let Q be a regular prime ideal of $R[X]_T$. Since $R[X]$ is a Marot ring, $Q = \Sigma fR[X]_T$, where f runs over all the regular elements of Q. Now the assumption implies that $Q = \Sigma fR[X]_T = \Sigma$
(A$_f[X]_T) = (\Sigma A_f)[X]_T$. Therefore Q is extended from R and hence from a prime ideal of R. □

Lemma 2.6 (cf. [?, Theorem 21.2] for the single variable case)
Let (R,T) be a PEP-pair. Then $R[X]_T$ is a Bezout ring.

Proof. Let $D = R[X]_T$ and f,g be regular elements of R[X]$.Then$h=f+gX$^{\deg f+1}$ is also a regular element of $R[X]$. So, by Lemma ??, $fD+gD = A_f D + A_g D = (A_f + A_g)D = A_h D = hD$. Hence R[X]$_T$ is a Bezout ring. □

Lemma 2.7 *Let R be an integral domain. If every prime ideal of $R[X]_T$ is extended from a prime ideal of R, then the saturation $\bar{T}$ of T is of the form $R[X] \setminus (\bigcup_\alpha P_\alpha[X])$, where P_α's are a collection of prime t-ideals of R.*

Proof. Let $f \in R[X] \setminus \bar{T}$. There exists a prime ideal Q_α of $R[X]$ such that $f \in Q_\alpha$ and Q$_\alpha \cap \bar{T} = \varnothing$. Shrinking Q_α, we may assume that Q_α is minimal over (f) so that Q_α is a t $-ideal. Now($Q$_\alpha)_T = P_\alpha[X]_T$ for some prime ideal P$_\alpha$ of R. Since Q_α is a t-ideal, so is P_α
. Now $\bar{T} = R[X] \setminus (\bigcup_\alpha Q_\alpha) = R[X] \setminus (\bigcup_\alpha P_\alpha[X])$. □

Remark Let R be an Marot ring with the property
(A)
. If we start with Q a regular prime t-ideal of $R[X]$, then $P = Q \cap R$
is also a regular prime t-ideal. So, for a domain, the complement of the
saturation of T can be realized as a union of prime t-ideals which are

extended from R. However this may not hold for arbitrary rings with zero divisors. The difficulty lies in that a zero divisor may not be contained in any proper (prime) t-ideal.

III Integral Domains

Lemma 3.1 *Let R be an integral domain. If each prime t-ideal of $R[X]_T$ is extended from a prime ideal of R, then each prime ideal of $R[X]_T$ is extended from a prime ideal of R.*

Proof. Let Q be a nonzero prime ideal of $R[X]_T$.
Then Q is a union of prime t-ideals, say $Q = \bigcup_\alpha Q_\alpha$. Now $Q = \bigcup_\alpha Q_\alpha = \bigcup_\alpha (P_\alpha[X]_T)$, where $P_\alpha = Q_\alpha \cap R$ is a prime ideal of R.
$\Rightarrow P := Q \cap R = \bigcup_\alpha (P_\alpha[X]_T) \cap R = \bigcup_\alpha (P_\alpha[X]_T \cap R) = \bigcup_\alpha P_\alpha$ is a prime ideal of R. $\Rightarrow Q \subseteq (\bigcup_\alpha P_\alpha)[X]_T \subseteq Q$. So $Q = (\bigcup_\alpha P_\alpha)[X]_T = P[X]_T$ is extended from a prime ideal P of R. $\square$

We will prove the main theorem.

Theorem 3.2 *Let R be an integrally closed domain and T a multiplicatively closed subset of $R[X]$ with the saturation $\bar{T}$ in $D[X]$, where $D = R[X]_T \cap K$. Then the following statements are equivalent.*

(1) *Each prime ideal of $R[X]_T$ is extended from R.*

(2) *$D = R[X]_T \cap K$ is a PVMD and $\bar{T} = N_v(D)$.*

(3) *$R[X]_T$ is a Bezout domain and $\bar{T} = N_v(D)$.*

(4) *$R[X]_T$ is a Prüfer domain and $\bar{T} = N_v(D)$.*

(5) *$R[X]_T$ is a PVMD and $\bar{T} = N_v(D)$.*

(6) *There exists a PVMD overring A of R such that $R[X]_T = A[X]_{N_v(A)}$.*

(7) *Each prime ideal of $R[X]_T$ is extended from a prime ideal of R.*

(8) *Each principal ideal of* $R[X]_T$ *is extended from* R

.

(9) $fR[X]_T = A_f[X]_T$ *for each element* f *of* $R[X]$

.

Proof. (??)⇒(??).

First we show that $\bar{T} \subseteq N_v(D)$. Let Q_T be a maximal t -ideal of $R[X]_T$. By [?, Lemma 3.17(1)], Q is a prime t-ideal. Put $P = Q \cap R$. By assumption $P \neq (0)$, and it can easily be shown that

P$isalsoa$t$-ideal.Now$Q=P[X]$.Then$R[X]$_Q = R[X]_{P[X]} = R_P(X)$ by [?, Lemma 2]. So

$$R[X]_Q \cap K = R_P. \tag{a}$$

Let $A = R[X]_T$ and recall that $A = \bigcap_{M \in \mathrm{Maxt}(A)} A_M$ [?, Proposition 2.8(3)]. Let $\Lambda = \{Q \mid Q$ is a prime ideal of R[X]$such that$QA$\in \mathrm{Maxt}(A)\}$. Then $\{QA \mid Q \in \Lambda\} = \mathrm{Maxt}(A)$. So

$$R[X]_T = \bigcap_{Q \in \Lambda} R[X]_Q. \tag{b}$$

From (??) and (??), $R[X]_T \cap K = (\bigcap_{Q\in\Lambda}$ R[X]$_Q) \cap K = \bigcap_{Q\in\Lambda}(R[X]_Q \cap K) = \bigcap_{P\in}$ $\Lambda \cap R R_P$, where $\Lambda \cap R = \{P \mid P = Q \cap R,\ Q \in \Lambda\}$. Put $D = \bigcap_{P\in\Lambda\cap R} R_P$. $\Rightarrow$ D=R[X]$_T \cap K$. $\Rightarrow D[X]_T \subseteq R[X]_T$. $\Rightarrow$ D[X]$_T = R[X]_T$. Thus $R \subseteq D$, $R[X]_T = D[X]_T$, and by Proposition ?? and Lemma ?? all prime ideals of $D[X]_T$ are extended from D. Now let f be an element of T. $\Rightarrow$ A$_f$ $\not\subset$ anyP$\in \Lambda \cap R$ since $P[X] \cap T = \varnothing$. $\Rightarrow$ Let $*$ be the $*$-operation on D induced by the overrings R_P [?, Theorem 32.5]. Then $(A_f D)_* = \bigcap_{P\in\Lambda\cap}$ R(A$_f R_P) = \bigcap_{P\in\Lambda\cap R} R_P = D$ $\Rightarrow$

$(A_f)_v = D$ [?, Theorem 34.1] $\Rightarrow f \in N_v(D)$
$\Rightarrow T \subseteq N_v(D)$. So the saturation $\bar{T}$ of T is
contained in $N_v(D)$. To prove the reverse containment we will first show
that each prime t-ideal Q_T of $D[X]_T$ is extended from a prime t
-ideal of D: Clearly, Q is a prime t-ideal of $D[X]$
[?, Lemma 3.17(1)]. By the assumption, $Q_T = I[X]_T$ for an ideal I
of D. So $Q_T = (Q_T)_t = (I_t[X]_T)_t$ [?, Lemma 3.4(3)]
$\Rightarrow Q_T = I_t[X]_T \Rightarrow P := Q_T \cap$
$D = I_t[X]_T \cap D = I_t$ since $T \subseteq N_v(D)$. $\Rightarrow$
$Q_T = I_t[X]_T = P[X]_T$ and P is a prime t-ideal. By Lemmas
Lem4 and ??, the saturation $\bar{T}$ of T in $D[X]$ is
$D[X] \setminus (\bigcup_{P\in\Gamma} P[X])$ for some collection Γ of
prime t-ideals of D. To show that $N_v(D) \subseteq \bar{T}$, let $f \in$
$N_v(D)$. Since each P appearing in the expression $\bar{T} = D[X]\setminus$
$(\bigcup_{P\in\Gamma} P[X])$ is a t-ideal of D, A_f
$\not\subset$
P$andso$f
$\notin$
$(\bigcup_{P\in\Gamma} P[X])$, i.e., $f \in \bar{T}$. So $N_v(D) \subseteq$
$\bar{T}$ and therefore $N_v(D) = \bar{T}$. From the information that
$D = \bigcap_{P\in\Lambda\cap R} R_P$ is integrally closed, that $\bar{T}$
$= N_v(D)$, and that every prime ideal of $D[X]_T = R[X]_T$ is extended
from D, we conclude that $R[X]_T = D[X]_T = D[X]_{\bar{T}} = D[X]_{N_v(D)}$
is a Bezout domain by Lemma ?? since PEP is always equivalent to
PEI. So $D = R[X]_T \cap K$ is a PVMD [?, Theorem 3.7], and $\bar{T}$ in
$D[X]$ coincides with $N_v(D)$.
The implication (??)$\Rightarrow$(??) is obvious.
Note that $R[X]_T = D[X]_T$. Then apply Lemma ??.
The implications (??)$\Rightarrow$(??) and (
Thm1New(4))$\Rightarrow$(??) are obvious.
For the implication (??)$\Rightarrow$(??), note
that $R[X]_T = D[X]_T$. By [?, Corollary 13] or
8, D is a PVMD.
(??)$\Rightarrow$(??). By the assumption each prime
ideal Q_T of $R[X]_T = A[X]_{N_v(A)}$ is extended from A
[?, Theorem 3.14]. So $Q_T \cap K \neq (0)$. $\Rightarrow Q_T \cap$
$R \neq (0) \Rightarrow Q_T$ is extended from R since R is
integrally closed [?, Lemma 3.4(3) and Theorem 4.1].
(??)$\Leftrightarrow$(??) is obvious.
(??)$\Rightarrow$(??) follows from Lemma ??.
(??)$\Rightarrow$(??) is a part of Lemma ??.
(??)$\Rightarrow$(??) is clear. □

In the proof of (??)$\Rightarrow$(??) of the above theorem, we proved

Corollary 3.3 (cf. [?, Theorem 4])
Let R be an integrally closed domain. Then the following are equivalent.

(1) *Each prime ideal of $R(X)$ is extended from R.*

(2) *R is a Prüfer domain.*

(3) *$R(X)$ is a Bezout domain.*

(4) *$R(X)$ is a Prüfer domain.*

(5) *Each ideal of $R(X)$ is extended from R.*

(6) *$fR(X) = A_f R(X)$ for each $f \in R[X]$.*

Corollary 3.4 *Let R be an integrally closed domain. If (R,T) is a PEP-pair, then $\bar{T} = N_v(D)$. In particular, $N_v(D) \subseteq \bar{T}$*
.

Remark In Theorem ??, (??) or (??) without $\bar{T} = N_v(D)$ does not imply (??). See the example in Section ??.

Extension of the Main Theorem to the Rings with Zero Divisors.]
Extension of the Main Theorem to the Rings with Zero Divisors.

Lemma 3.1 *Let D be a ring, S a saturated multiplicative subset of $D[X]$*
, and f a regular element of D[X] which is not in S. Then there exists a regular prime t-ideal Q of $D[X]$ such that $f \in Q \subseteq$ D[X]$\backslash S$.

Proof. Let B be the set $\{J \mid f \in J, J\cap$ S=$\varnothing$, and J is a t-ideal of $D[X]\}$. Since $fD[X] \in S$, S$\neq \varnothing$ (note that $(f) \cap S = \varnothing$ because S is saturated). Let $\{J_\alpha\}_\alpha$ be a chain in S, and let J=$\bigcup_\alpha J_\alpha$. Clearly J belongs to S. By Zorn's lemma S has a maximal element, say Q. To show that Q is a prime ideal, let g,h *be elements of* D[X] *such* fg$\in Q$ while neither of the two is contained in Q. By the maximality of Q, we can choose $t_1 \in$ (Q,g)$_t \cap S$, $t_2 \in (Q,h)_t \cap S$. Then $t_1 t_2 \in$ ((Q,g)$_t(Q,h)_t)_t \cap S = ((Q,g)(Q,h))_t \cap S \subseteq Q_t \cap$ S=Q$\cap S$, which contradicts that $Q \cap S = \varnothing$. So Q is a prime ideal. □

We slightly strengthen a part of [?, Theorem 22.5] which is stated for
an additively regular v-ring with property
(A)
 by replacing this by a Marot ring with property
(A)
. Note that a v-ring (see [?, ?] for the definition) is always integrally closed and an additively regular ring is necessarily a Marot ring.

Proposition 3.2 (cf. [?, Theorems 3.7, 3.14, and Corollary 3.16].)
Let R be an integrally closed Marot ring with property
(A)
. Then the following statements are equivalent.

(1) $(R, N_v(R))$ *is a PEP-pair.*

(2) $(R, N_v(R))$ *is a PEI-pair.*

(3) $(R, N_v(R))$ *is an IEI-pair.*

(4) R *is a Prüfer v-multiplication ring.*

(5) $R[X]_{N_v(R)}$ *is a Prüfer v*
-multiplication ring.

(6) $R[X]_{N_v(R)}$ *is a Prüfer ring.*

(7) $R[X]_{N_v(R)}$ *is a Bezout ring.*

Proof. The implication (??)⇒(
Pro4(7)) follows from Lemma ??. The implications (??)
⇒(??)⇒(??) and (??)
⇒(??)⇒(??) are obvious. For
the implication (??)⇒(??), modify the
proof for the domain case [?, Theorem 3.1]. (??)⇒
(??).*Let*I*beafinitelygeneratedregularidealof*R.*Then*

(1)=((IR[X]$_{N_v(R)})(IR[X]_{N_v(R)})^{-1})_t = ((II^{-1})[X]_{N_v(R)})_t = (II^{-1})_t[X]_{N_v(R)}$
by Proposition ??. Hence $(1) = (II^{-1})_t$. □

Now we are ready for the zero-divisor version of the main theorem of the
first part. Although the proof is about the same as that for the domain

case, for future reference and for the sake of completeness we include it.
Huckaba and Papick [?, Lemma 3.1] proved that for an integrally closed Marot ring with property (A) and a multiplicatively closed set T of $R[X]$ such that $T \cap \mathbf{Z}$ (R[X])=∅, if each regular prime ideal of $R[X]_T$ is extended from R, then $R[X]_T$ is a Bezout domain. They used the u-or-u $^{-1}$ version for the ring with zero divisors. Through a different approach, we give another proof of their result and a complete characterization of rings R and multiplicative sets T such that $\mathrm{Spec}(R[X]_T) = \mathrm{Spec}(R)^e$.

Theorem 3.3 *Let R be an integrally closed Marot ring with the property* (A) *and T a multiplicatively closed subset of $R[X]$ with the saturation $\bar{T}$ in $D[X]$, where $D = R[X]_T \cap K$. Then the following statements are equivalent.*

(1) *Each regular prime ideal of $R[X]_T$ is extended from R.*

(2) *$D = R[X]_T \cap K$ is a PVMR and $\bar{T} = N_v(D)$.*

(3) *$R[X]_T$ is a Bezout ring and $\bar{T} = N_v(D)$.*

(4) *$R[X]_T$ is a Prüfer ring and $\bar{T}$ =$N_v(D)$.*

(5) *$R[X]_T$ is a PVMR and $\bar{T} = N_v(D)$.*

(6) *There exists a PVMR overring A of R such that $R[X]_T = A[X]_{N_v(A)}$.*

(7) *Each regular prime ideal of $R[X]_T$ is extended from a prime ideal of R.*

(8) *Each regular principal ideal of $R[X]_T$ is extended from R.*

(9) *$fR[X]_T = A_f[X]_T$ for each regular element f of $R[X]$.*

Proof. For brevity of notation, given a multiplicative subset M of a ring A we will denote the regular elements of M by (M)
and the ring $A_{(M)}$ by A_M. For a prime ideal P of R and a prime ideal Q of $R[X]$, we also use the notation $R_P = R_{(R\setminus P)}$, and $R[X]_Q = R[X]_{(R[X]\setminus Q)}$. Thus $(R_P[X])_S = (R_{(R}$
$\setminus P)[X])_{(S)}$, where S is a multiplicative subset of $R[X]$.
(??)$\Rightarrow$(??). First we show that $\bar{T}$
$\subseteq N_v(D)$. Let Q_T be a maximal t-ideal of $R[X]_T$. By Proposition ??(??), Q is a prime t-ideal. Put
P=Q$\cap R$. By assumption $P \neq (0)$, and it can easily be shown that P is also a t-ideal. Put $S = R[X] \setminus Q$ and $H = R_P$. Then
R[X]$_Q = R_P[X]_S = H[X]_S$ and $S \subseteq N(H)$. So $H[X]_S \cap K = H$
, i.e.,

R[X]$_Q \cap K = R_P$.
(a)

Let $A = R[X]_T$ and recall that $A = \bigcap_{M\in \mathrm{Maxt}(A)} A_{(M)}$
by Proposition ??. Let $\Lambda = \{Q \mid Q$ is a prime ideal of
R[X] $such that$QA$\in \mathrm{Maxt}(A)\}$. Then $\{QA \mid Q \in \Lambda\} =$
$\mathrm{Maxt}(A)$. So

R[X]$_T = \bigcap_{Q\in\Lambda} R[X]_Q$.
(b)

From (??) and (??), $R[X]_T \cap K = (\bigcap_{Q\in}$
$\Lambda R[X]_Q) \cap K = \bigcap_{Q\in\Lambda}(R[X]_Q \cap$
K)=$\bigcap_{P\in\Lambda\cap R} R_P$, where $\Lambda \cap R = \{P \mid$
P=Q$\cap R$, $Q \in \Lambda\}$. Put $D = \bigcap_{P\in\Lambda\cap R} R_P$.
Then $D = R[X]_T \cap K$. So $D[X]_T \subseteq R[X]_T$. So
D[X]$_T = R[X]_T$. Thus $R \subseteq D$, $R[X]_T = D[X]_T$, and all prime ideals of $D[X]_T$ are extended from D (Proposition ?? and Lemma ??). Now let f be an element of T. $\Rightarrow A_f$
$\not\subseteq$
anyP$\in \Lambda \cap R$ since $P[X] \cap T = \varnothing$. $\Rightarrow$
Let $*$ be the $*$-operation on D induced by the overrings R_P. Then

(A$_f D)_* = \bigcap_{P\in\Lambda\cap R}(A_f R_P) = \bigcap_{P\in\Lambda}$

$\cap RR_P = D \Rightarrow (A_f)_v = D$ [?, Proposition 32(iv)]
$\Rightarrow f \in N_v(D) \Rightarrow T \subseteq N_v(D)$. So
the saturation $\bar{T}$ of T is contained in $N_v(D)$. To prove the reverse containment we will first show that each regular prime t-ideal
Q_T of $D[X]_T$ is extended from a regular prime t-ideal of D: Clearly, Q can be assumed to be a regular prime t-ideal of $D[X]$. By
the assumption, $Q_T = I[X]_T$ for an ideal I of D. So
$Q_T = (Q_T)_t = (I[X]_T)_t = (I_t[X]_T)_t$ by Proposition
Pro2 $\Rightarrow Q_T = I_t[X]_T \Rightarrow P := Q_T \cap$
D=$I_t[X]_T \cap D = I_t$ since $T \subseteq N_v(D)$. $\Rightarrow$
$Q_T = I_t[X]_T = P[X]_T$ and P is a regular prime t-ideal. To show that $N_v(D) \subseteq \bar{T}$, let $f \in N_v(D)$. We will show this by contraposition. Suppose f
$\notin$
$\bar{T}$. By Lemma ??, there exists a prime t-ideal Q of $D[X]$ such that $f \in Q \subseteq D[X] \setminus \bar{T}$. The previous argument guarantees that Q is extended from a prime t-ideal P of D. So
A_f
$\not\subset$
P, $i.e.,$f
$\notin$
P[X]=Q, $contrary to that$ f$\in Q$. So $N_v(D) \subseteq \bar{T}$ and therefore $N_v(D) = \bar{T}$. From the information that $D = \bigcap_{P \in}$
$\Lambda \cap RR_P$ is integrally closed, that $\bar{T} = N_v(D)$, and that every prime ideal of $D[X]_T$ is extended from D, we conclude that
R[X]$_T = D[X]_T = D[X]_{\bar{T}} = D[X]_{N_v(D)}$ is a Bezout ring by Lemma
?? since PEP is always equivalent to PEI and any overring of a Marot
ring with property
(A)
is always a Marot ring with property
(A)
[?, Corollaries 2.6 and 7.3]. So $D = R[X]_T \cap K$ is a PVMR by Proposition ??, and $\bar{T}$ in $D[X]$ coincides with $N_v(D)$.
The implication (??)⇒(??) is obvious:
note that $R[X]_T = D[X]_T$; then apply Proposition ??.
The implications (??)⇒(??) and (
Thm2New(4))⇒(??) are obvious.
For the implication (??)⇒(??), note
that $R[X]_T = D[X]_T = D[X]_{\bar{T}} = D[X]_{N_v(D)}$, and then apply Proposition ??.
(??)⇒(??). The assumption and Proposition
?? imply that each prime ideal Q_T is extended from A. So

$\mathtt{Q}_T \cap K \neq (0)$. $\Rightarrow$ $Q_T \cap R \neq (0)$ $\Rightarrow$
$\mathtt{Q}_T$ is extended from R since R is integrally closed (Proposition Pro3 and Lemma ??).
(??)$\Leftrightarrow$(??) is obvious.
(??)$\Rightarrow$(??) follows from Lemma ??.
(??)$\Rightarrow$(??) is a part of Lemma ??.
(??)$\Rightarrow$(??) is clear. □

In the proof of (??)$\Rightarrow$(??) of the above theorem, we proved

Corollary 3.4 *Let (R,T) be a PEP-pair. Then $\bar{T} = N_v(D)$. In particular, $N_v(D) \subseteq \bar{T}$.*

Consideration without the assumption 'integrally closed']
Consideration without the assumption 'integrally closed'
Unless otherwise specified, we denote by R'
the integral closure of a ring R in its total quotient ring.
Although
[?, Theorem 4.2] is stated differently, its proof given implies the
following.

Proposition 3.1 *Let $R \subseteq D$ be an integral ring extension and T a multiplicative subset of $R[X]$*
(
not necessarily $T \cap \mathbf{Z}(R[X]) = \varnothing$
)
. *Then*
$\mathrm{Spec}(R[X]_T) = \mathrm{Spec}$
(R)^e $\Leftrightarrow \mathrm{Spec}(D[X]_T) = \mathrm{Spec}(D)^e$.

We excerpt a result from [?, Theorem 3(a)].

Lemma 3.2 *Let $R \subseteq D$ be an integral ring extension. Then $N(R)' = N(D)$, where $N(R)'$ is the saturation of $N(R)$ inD[X].Inparticular,N(R)$' = N(R')$.*

From Proposition ?? and Lemma ??, we immediately deduce

Theorem 3.3 *Let L be a field containing R and R' be the integral closure of R in L. Then $\mathrm{Spec}\, R(X) =$ (R)^e $\Leftrightarrow R'$ is a Prüfer domain.*

Recall that every overring of a Marot ring is a Marot ring and each overring
of a ring with property

(A)
has also property
(A)
[?, Corollaries 2.6 and 7.3].

Theorem 3.4 *Let* R *be a Marot ring with property*
(A)
. *Then* $\operatorname{Spec} R(X) = \operatorname{Spec}(R)^e \Leftrightarrow R'$
is a Prüfer ring.

Proof. Applying Proposition ??, Lemma ??,
and Theorem ?? successively, we obtain the equivalences:
$(R, N(R))$
is a PEP-pair $\Leftrightarrow (R', N(R))$ is a PEP-pair
$\Leftrightarrow (R', N(R)')$ is a PEP-pair $\Leftrightarrow$
(R$'$, $N(R')$) is a PEP-pair $\Leftrightarrow R'$ is
a Prüfer ring. □

Theorem 3.5 *Let* R *be a Marot ring with property*
(A)
. *Then* $\operatorname{Spec}(R[X]_T) = \operatorname{Spec}(R)^e \Leftrightarrow$
D=R$'[X]_T \cap K$ *is a PVMR and* $T' = N_v(D)$.

Proof. Applying Proposition ?? and Theorem
Thm2 successively, we obtain the equivalences: $\operatorname{Spec}(R[X]_T) =$
$\operatorname{Spec}(R)^e \Leftrightarrow (R, T)$ is a PEP-pair $\Leftrightarrow$
(R$'$, T) is a PEP-pair $\Leftrightarrow D = R'[X]_T \cap K$
is a PVMR and $T' = N_v(D)$. □

Corollary 3.6 *Let* R *be a Marot ring with property*
(A)
. *Then* $\operatorname{Spec}(R[X]_{N_v(R)}) = \operatorname{Spec}(R)^e \Leftrightarrow$
D=R$'[X]_{N_v(R)} \cap K$ *is a PVMR and* $N_v(R)' = N_v(D)$
.

Corollary 3.7 *Let* R *be a Noetherian ring. Then* Spec
(R[X]$_{N_v(R)}) = \operatorname{Spec}(R)^e \Leftrightarrow N_v(R)'$
=N$_v(R')$.

Proof. It is well-known that a Noetherian ring is a Marot
ring with property
(A)
([?, Theorem 7.2] and [?, Theorem 82]) and that the
integral closure of a Noetherian ring is a Krull ring [?, Theorem
10.1]
and hence a PVMR. The proof is completed if we show that R'
[X]$_{N_v(R)} \cap K = R'$. Let $k \in R'[X]_{N_v(R)} \cap$

K$\Rightarrow kf \in R'$ for $f \in N_v(R) \Rightarrow$
k$A_f R \subseteq R' \Rightarrow kA_f I \subseteq I$ for a finitely
generated regular ideal I of $R \Rightarrow kI_v \subseteq$
(k$I_v)_v = (k(A_f)_v I_v)_v = (kA_f I)_v \subseteq I_v \Rightarrow$
k$I_v \subseteq I_v$. Since I_v is a finitely generated regular ideal
of R, $k \in R'$. Hence $R'[X]_{N_v(R)} \cap K \subseteq$
R$'$ and therefore $R'[X]_{N_v(R)} \cap K = R'$.

□

In [?, Theorem 3.1], Houston and Zafrullah showed that for a
Noetherian
domain R, $\mathrm{Spec}(R[X]_{N_v(R)}) = \mathrm{Spec}$
(R)$^e \Leftrightarrow$
t-dim
(R)≤ 1. Combining this with the preceding Corollary, we obtain

Corollary 3.8 *Let R be a Noetherian domain. Then the following statements are equivalent.*

(1) $\mathrm{Spec}(R[X]_{N_v(R)}) = \mathrm{Spec}$
(R)^e.

(2)

t-dim
(R)≤ 1.

(3) $N_v(R)' = N_v(R')$.

Remark If R is a Noetherian domain with
t-dim
(R)=$1, so are$R[X]andR$[\![X]\!]$ [?, Corollary 3.8].

Although $N(R)' = N(R')$ as is indicated in Lemma ??
, we will give an example such that $N_v(R)' \neq N_v(R'$
) $.This example also shows that the condition$T$' = N_v(D)$ is
indispensable in Theorem ??.

Example 3.9 (1) There exists a domain R and a multiplicative set
T of $R[X]$ such that $D = R'[X]_T \cap K$ is a PVMD while
T$' \neq N_v(D)$. So (2) $D = R'[X]_T \cap K$ is a PVMD
$\not\Rightarrow$
(R,T)$is a PEP-pair. (3) There exists a domain$R$such that$
N$_v(R') \supsetneq N_v(R)'$, where $N_v(R)'$
is the saturation of $N_v(R)$ in $R'[X]$.

Demonstration. Let R be a 2-dimensional Noetherian local domain which is not Macaulay and $T = N_v(R)$. Then $D = R'$ [X]$_{N_v(R)} \cap K = R'$ is a Krull domain and hence a PVMD. However, $N_v(R) = N(R)$ (see [?, Ex. 2 on page 102]). So $N_v(R)' = N(R)' = N(R')$. Note that all maximal ideals of R' have height 2 and all maximal t-ideals of R' have height 1. Then $\bigcup\{M[X] \mid M \in \mathrm{Max}(R')\} \not\subseteq \bigcup\{P[X] \mid P \in X^1(R')\}$. For otherwise $M[X] \subseteq \bigcup P[X] \Rightarrow M[X] \subseteq$ some $P[X] \Rightarrow M \subseteq$ some $P \in X^1(R')$, a contradiction. Thus $N_v(R') = R'[X] \setminus \bigcup\{P[X] \mid P \in X^1(R')\} \not\subseteq N(R') = R' \setminus \bigcup\{M[X] \mid M \in \mathrm{Max}(R')\}$. Hence $N_v(R') \supsetneq N(R')$ and therefore $N_v(R') \supsetneq N(R') = N_v(R)'$.

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