Krull domain, power series ring, valuation domain
,$X_n]\!]_{D-(0)}$ is quasilocal and in this case $D[\![X_1, \ldots$
,$X_n]\!]_{D-(0)}$ is actually an n-dimensional regular local ring. We also show that if D is an SFT Prüfer domain, then $D[\![\{$
$X_\alpha]\!]_{1_{D-(0)}}$ is a Krull domain for any set of indeterminates $\{X_\alpha\}$.

ANTI-ARCHIMEDEAN RINGS AND POWER SERIES RINGS

D. D. ANDERSON, B. G. KANG, AND M. H. PARK

Abstract. We define an integral domain D to be anti-Archimedean if

$\bigcap_{n=1}^{\infty} a^n D \neq 0$ for each $0 \neq a \in D$. For example, a valuation domain or SFT Prüfer domain is anti-Archimedean if and only if it has no height-one prime ideals. A number of constructions and stability results for anti-Archimedean domains are given. We show that D is anti-Archimedean $\Leftrightarrow$ $D[\![X_1, \ldots$

1. Introduction

Throughout this paper, D will denote an integral domain with quotient field K.
An integral domain D is *Archimedean* if $\bigcap_{n=1}^{\infty} d^n D = 0$ for each nonunit d of D. Examples of Archimedean domains include domains satisfying the ascending chain condition on principal ideals (ACCP), domains in which every nonunit is contained in a height-one prime ideal, and domains which are completely integrally closed. In this paper we will be concerned with integral domains that are as far as possible from being Archimedean. We define D to be *anti-Archimedean* if $\bigcap_{n=1}^{\infty} d^n D \neq 0$ for each nonzero $d \in D$. Perhaps the first example of an anti-Archimedean domain that comes to mind is a valuation domain with no height-one prime ideal.
Section **??** gives some basic results concerning anti-Archimedean domains. For example, D is anti-Archimedean $\Leftrightarrow$ K is the complete integral closure of D $\Leftrightarrow$ each nonzero prime ideal of D contains a bounded element. If D is anti-Archimedean, every nonzero prime ideal of D has infinite height. While the converse is true for valuation domains or SFT-Prüfer domains, we remark that an example of Nakayama shows that the converse is false even for Bezout domains.
We show that an integral domain that is an algebraic extension of an anti-Archimedean domain is again anti-Archimedean. Also, an ascending union or locally finite intersection of anti-Archimedean domains is anti-Archimedean. Finally, $D(X)$ is

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anti-Archimedean if and only if D is anti-Archimedean and the integral closure $\bar{D}$ of D is Prüfer.

In Section **??** we consider power series rings over anti-Archimedean domains. We show that D is anti-Archimedean if and only if $D[\![\{X_\alpha\}$ $]\!]_{i_{D-(0)}}$ is quasilocal where $\{X_\alpha\}$ is a nonempty set of power series indeterminates (for the definition of the three types of power series rings $D[\![\{X_\alpha\}]\!]_i$ (i=1,2,3)$when\{X_\alpha\}$ is infinite, see G2). Perhaps the most interesting result of this paper is Theorem Thm1.2 which states that if D is anti-Archimedean, then D$[\![$X$_1, \ldots, X_n]\!]_{D-(0)}$ is an n-dimensional regular local ring. As an immediate corollary, we get that for any set of indeterminates $\{X_\alpha\}$, $D[\![\{X_\alpha\}$ $]\!]_{1_{D-(0)}}$ is a UFD when D is anti-Archimedean. Finally, we show that if D is an SFT Prüfer domain, $D[\![\{X_\alpha$ $]\!]_{1_{D-(0)}}$ is a Krull domain.

We follow standard notation and terminology from commutative ring theory, see [**?**], [**?**], or [**?**]. Throughout D will denote an integral domain with quotient field K. Spec(D) denotes the set of prime ideals of D and Max(D) (resp., $X^{(1)}(D)$) denotes the set of maximal ideals of D (resp., height-one prime ideals of D).$We use\bar{D}$ to denote the integral closure of D.

2. Anti-Archimedean Integral Domains

Let D be an integral domain with quotient field K. An element $d \in D$ is *Archimedean* if $\bigcap_{n=1}^{\infty} d^n D = 0$ and d is *non-Archimedean* or *bounded* if d is not Archimedean, that is, $\bigcap_{n=1}^{\infty} d^n D \neq 0$. If $d \in$ D$is Archimedean, so is$sd$for any$s$\in D$, but a factor of an Archimedean element need not be Archimedean (for example, let V be a two-dimensional valuation domain with prime ideals $0 \subsetneq$ P$\subsetneq M$ and consider sd where $d \in P - \{0\}$ and $s \in$ M-P)$.The set$B(D) $of bounded elements of$D$is easily seen to be a$ saturated multiplicatively closed set.

The integral domain D is *Archimedean* if each nonunit of D is Archimedean and D is *anti-Archimedean* if each nonzero element of D is bounded. Note that D is both Archimedean and anti-Archimedean if and only if D is a field. It is well-known that if D satisfies ACCP or is completely integrally closed, then D is Archimedean. A Bezout domain, or more generally a GCD domain, is Archimedean if and only if it is completely integrally closed. If P is a height-one prime ideal of an integral domain D, then $\bigcap_{n=1}^{\infty} a^n D = 0$ for each $a \in P$ ([**?**, Theorem 1] or [**?**, Corollary 1.4]). For results concerning Archimedean domains, see [**?**].

Proposition 2.1. *If an integral domain D is anti-Archimedean, then every nonzero prime ideal of D has infinite height* (*or* *equivalently, D has no height-one prime ideals). For a valuation domain, the converse is true.*

Proof. Suppose that D is anti-Archimedean. If D contains a nonzero prime ideal P with $\operatorname{ht} P < \infty$, then P contains a prime ideal P_0 with $\operatorname{ht} P_0 = 1$. But then by the remark in the previous paragraph, $\bigcap_{n=1}^{\infty} a^n D = 0$ for each $a \in P_0$. Let V be a valuation domain with no height-one prime ideal and let $0 \neq a \in V$. Then there is a prime ideal P with $0 \neq P \subsetneq aV$ and hence $P \subseteq \bigcap_{n=1}^{\infty} a^n V$.

□

Example 2.2. Nakayama [?] gave an example of a completely integrally closed (and hence Archimedean) Bezout domain D with no rank-one valuation domain overrings. Thus while D is not anti-Archimedean, every valuation overring of D is anti-Archimedean. Also, D has no nonzero prime ideals of finite height. Thus the converse of the first statement of Proposition **??** does not hold and the second statement cannot be extended to Bezout domains. Finally, note that each localization D_P of D (P a prime ideal) is anti-Archimedean.

We next show that for a certain class of Prüfer domains, the converse of the first statement of Proposition **??** does hold. Recall [?, ?, ?, ?] that an ideal A of a ring R is called an
SFT-ideal
SFT-ideal provided there exists a finitely generated ideal $B \subseteq A$ and a natural number k so that $a^k \in B$ for each $a \in A$. Then R is called an SFT-ring *provided each ideal of R is an SFT-ideal.*

Proposition 2.3. *Let D be an SFT Prüfer domain. Then D is anti-Archimedean if and only if $X^{(1)}(D) = \varnothing$.*

Proof. Proposition **??** gives one direction. Conversely, let D be an SFT Prüfer domain with $X^{(1)}(D) = \varnothing$. Let a be a nonzero nonunit of D. Then $\sqrt{(a)} = Q_1 \cap \cdots \cap Q_m$ where { $Q_1, \ldots, Q_m$ is the finite ([?, Corollary 2.6]) set of prime ideals minimal over (a). Since $X^{(1)}(D) = \varnothing$, there is a nonzero prime ideal P_i with $P_i \subseteq \bigcap_{n=1}^{\infty}$ Q_i^n. Since for some natural number k, we have $(Q_1 \cap \ldots$ $\cap Q_m)^k \subseteq (a)$, $\bigcap_{n=1}^{\infty}(Q_1 \cap \cdots \cap$ $Q_m)^n = \bigcap_{n=1}^{\infty}(a^n)$. Since Q_i and Q_j $(i \neq$ j)$areco-maximal$,$\bigcap_{n=1}^{\infty}(a^n) = \bigcap_{n=1}^{\infty}$ $(Q_1 \cap \cdots \cap Q_m)^n = \bigcap_{n=1}^{\infty}(Q_1^n \cap$ $\ldots \cap Q_m^n) \supseteq P_1 \cap \cdots \cap P_m \neq 0$.

□

Proposition 2.4. *An integral domain D with quotient field K is anti-Archimedean if and only if K is the complete integral closure of D.*

Proof. Note that for $0 \neq d \in D$, $\frac{1}{d}$ is almost integral over D $\Leftrightarrow \bigcap_{n=1}^{\infty} d^n D \neq 0$. Thus D is

anti-Archimedean $\Leftrightarrow$ each $\frac{1}{d}$ $(0 \neq d \in D$
$)$ is almost integral over $D \Leftrightarrow K$ is the complete
integral closure of D.

□

It immediately follows from Proposition **??** *that an overring of an anti-Archimedean domain is anti-Archimedean. However, the next proposition gives a stronger result. Note that the integral domain D[X] $isneveranti-Archimedean$.*

Proposition 2.5. *Let $D \subseteq S$ be an extension of integral domains with S algebraic over D. If D is anti-Archimedean, then so is S.*

Proof. Let $0 \neq s \in S$. Since $D \subseteq S$ is algebraic, $sS \cap D \neq 0$. Let $0 \neq r \in sS \cap D$. Then $0 \neq \bigcap_{n=1}^{\infty} r^n D \subseteq \bigcap_{n=1}^{\infty} s^n S$.

□

If the pair of domains $D \subseteq S$ are close enough together, we can descend.

Proposition 2.6. *Let $D \subseteq S$ be a pair of integral domains with [D:S]$\neq 0$. Then D is anti-Archimedean if and only if S is anti-Archimedean.*

Proof. If D is anti-Archimedean, so is its overring S by Proposition **??**. Conversely, suppose that S is anti-Archimedean. Let $0 \neq d \in D$ and choose $0 \neq d_0 \in D$ with $d_0 S \subseteq D$. Then $0 \neq \bigcap (dd_0)^n S \subseteq \bigcap d^n D$.

□

Example 2.7. Let (V, M) be a valuation domain of the form $V = K + M$ $(K$ a field) where $X^{(1)}(V) = \varnothing$. Let D_0 be a subring of K;
then $D = D_0 + M$ is anti-Archimedean since $[D : V] = M$. By choosing D_0 to be (non-)quasilocal we get (non-)quasilocal anti-Archimedean domains. By choosing D_0 to be (non-)integrally closed in K, we get D to be (non-)integrally closed. Finally, with an appropriate choice of D_0 we can make D Prüfer or non-Prüfer.

Proposition 2.8. *Let D be an integral domain. Then the following statements are equivalent:* (1)
$Disanti-Archimedean,$
) every nonzero prime ideal of D contains a bounded element, (2) *every*
$(nonzero)$ *prime ideal of D is an anti-Archimedean ring, and*
)(3
) every maximal ideal of D is an
anti-Archimedean ring.

Proof. (**??**) $\Rightarrow$ (**??**). Clear. (**??**)
$\Rightarrow$ (**??**). Let B be the set of bounded elements of
D; $soBisasaturatedmultiplicativelyclosedsubsetof$D.$Suppose$

there exists $0 \neq d \in D - B$. Then $(d) \cap B = \varnothing$, so (d) can be enlarged to a prime ideal P with $P \cap B = \varnothing$, a contradiction. Thus D is anti-Archimedean. (**??**) $\Rightarrow$ (**??**). Let P be a nonzero prime ideal of D and let $0 \neq p \in P$. By hypothesis, $\bigcap_{n=1}^{\infty} p^n D \neq 0$. But $\bigcap_{n=1}^{\infty} p^{n+1} D \subseteq \bigcap_{n=1}^{\infty} p^n P$, so P$isanti-Archimedean.$(**??**)$\Rightarrow$ (Pro1.8(4)). Clear. (**??**) $\Rightarrow$ (**??**). Let $0 \neq d \in D$. We need $\bigcap_{n=1}^{\infty} d^n D \neq 0$. We can assume d $isanonunit, so$d$\in M$ for some maximal ideal M. Then $0 \neq \bigcap_{n=1}^{\infty} d^n M \subseteq \bigcap_{n=1}^{\infty} d^n D$.

□

Corollary 2.9. *Let D_1 and D_2 be integral domains which are subrings of a field K. Suppose that D_1 and D_2 have the same set of maximal ideals. Then D_1 is anti-Archimedean if and only if D_2 is anti-Archimedean.*

For examples of integral domains with the same set of maximal ideals, see [**?**]. *We end this section with four more ways to construct new anti-Archimedean domains from old ones.*

Proposition 2.10. *Let K be a field and let $\{D_\alpha\}$ be a directed system of subrings of K each of which is anti-Archimedean. Then $D = \bigcup_\alpha D_\alpha$ is anti-Archimedean.*

Proof. Let K_α be the quotient field of D_α. Then $\mathrm{K}_0 = \bigcup_\alpha K_\alpha$ is the quotient field of D. We use Proposition **??**. Now each $D_\alpha \subseteq K_\alpha$ is almost integral. Thus $D = \bigcup_\alpha D_\alpha \subseteq \bigcup_\alpha K_\alpha = K_0$ is almost integral. So D is anti-Archimedean.

□

Proposition 2.11. *Let D be an anti-Archimedean domain with quotient field K. Let k be a subfield of K with the property that $dD \cap k \neq 0$ $foreach 0 \neq d \in D$. Then $D \cap k$ is an anti-Archimedean domain with quotient field k.*

Proof. Again we use Proposition **??**. Let $x \in k$. Then there is a $0 \neq$ d$\in D$ with $dx^n \in D$ for every $n \geq 1$. Now by hypothesis, there is a $0 \neq d_0 \in dD \cap k$. Then $d_0 x^n \in D \cap k$ for each $n \geq 1$. This shows that k is the quotient field of $D \cap k$ and that k is almost integral over $D \cap k$. Thus $D \cap k$ is anti-Archimedean.

□

Example 2.12. (An anti-Archimedean valuation domain W with $W \cap$ k$notanti-Archimedean.)Let$(V,M)$beavaluationdomainwithset$ of prime ideals $M = P_0 \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots$ where $\bigcap_{n=1}^{\infty} P_n = 0$ and $V/M = \mathbb{Q}$, the rational numbers. So V is anti-Archimedean. Let ϕ:

V→ $V/M=\mathbb{Q}$ be the natural map and let $W=\phi^{-1}($ Z$_{(p)})$ where $p>0$ is prime. Then W is a valuation domain with maximal ideal M', $W/M\cong\mathbb{Z}_{(p)}$, and W$/M'\cong\mathbb{Z}/p\mathbb{Z}$. So p is a nonunit in W. Thus $W\cap\mathbb{Q}=\mathbb{Z}_{(p)}$. So W is anti-Archimedean while $W\cap\mathbb{Q}$ is not anti-Archimedean.

Proposition 2.13. *Let D be an integral domain with quotient field K and let $\{D_\alpha\}$ be a collection of overrings of D so that D=$\bigcap_\alpha D_a$ is locally finite. Then D is anti-Archimedean if and only if each D_α is anti-Archimedean.*

Proof. Certainly if D is anti-Archimedean, so is each overring D_α $.Conversely, suppose that each$D$_\alpha$ is anti-Archimedean. Let $0\neq d\in D$. Then d is a unit in each D_α, say, except for some $\alpha_1,\ldots,\alpha_n$. Then each $\bigcap_{n=1}^\infty$ d$^n D_{\alpha_i}\neq 0$, so there is a $0\neq d_i\in D_{\alpha_i}$ with $d_iD_{\alpha_i}\subseteq\bigcap_{n=1}^\infty d^nD_\alpha$ $_i$. By suitable multiplication by an element of D, we can assume that each $0\neq d_i\in D$. Put $d_\alpha=1$ for $\alpha\neq\alpha$ $_1,\ldots,\alpha_n$. Then $\bigcap_{n=1}^\infty$ d$^nD=\bigcap_{n=1}^\infty d^n(\bigcap_\alpha D_\alpha)$ $=\bigcap_{n=1}^\infty(\bigcap_\alpha d^nD_\alpha)$ $=\bigcap_\alpha(\bigcap_{n=1}^\infty d^nD_\alpha)\supseteq$ $\bigcap_\alpha d_\alpha D_\alpha\supseteq d_1\cdots d_nD\neq 0$.

□

Remark 2.14. Example **??** shows that Proposition **??** may fail if the intersection is not locally finite.

For our last result of this section, recall that for a domain D, $D(X)$ $=\{f/g\mid f,g\in D[X]$, the coefficients of g generate $D\}$.

Theorem 2.15. *Let D be an integral domain. Then $D(X)$ is anti-Archimedean if and only if D is anti-Archimedean and the integral closure $\bar{D}$ of D is a Prüfer domain.*

Proof. (⇒) Suppose that $D(X)$ is anti-Archimedean. Let $0\neq$ d$\in D$. Then $(\bigcap_{n=1}^\infty d^nD)D(X)=\bigcap_{n=1}^\infty$ d$^nD(X)\neq 0$, so $\bigcap_{n=1}^\infty d^nD\neq 0$. Thus D is anti-Archimedean. Recall that $\bar{D}$ is a Prüfer domain if and only if every nonzero prime ideal Q of $D(X)$ is extended from D ([**?**, Theorem 19.15] and [**?**, Lemma 4.2]). Let $P=Q\cap R$. Suppose $Q\neq P(X)$. Choose $f\in Q\cap D[X]-P[X]$. By [**?**, Theorem A] there exists a prime ideal Q_0 of $D[X]$ with $f\in Q_0\subseteq Q\cap$ D[X]andQ$_0\cap D=0$. So $\operatorname{ht}Q_0=1$ and hence ht Q$_0D(X)=1$. But this contradicts the hypothesis that $D(X)$ is anti-Archimedean.
(⇐) Let $\bar{D}$ be a Prüfer domain; so every prime ideal of $D(X)$ is extended from D. Let Q be a nonzero prime ideal of $D(X)$

, *so*Q=P(X) *where*P=Q∩D. Let $0 \neq d \in P$. Since D is anti-Archimedean, $0 \neq \bigcap_{n=1}^{\infty} d^n D \subseteq \bigcap_{n=1}^{\infty} d^n D(X)$. Thus Q contains a bounded element. By Proposition **??**, $D(X)$ is anti-Archimedean.

□

3. Power Series Rings

Let D be an integral domain with quotient field K. While the polynomial ring $D[X_1, \dots, X_n]_{D-(0)} = K[X_1, \dots, X_n]$ is reasonably well understood, the same cannot be said for $D[\![X_1, \dots ,X_n]\!]_{D-(0)}$. Arnold [**?**, Lemma 3.5 and Proposition 4.5] *showed that $D[\![X_1, \dots, X_n]\!]_{D-(0)}$ is an n-dimensional Noetherian Krull domain if D is a finite-dimensional SFT Prüfer domain. Motivated by this, Kang and Park* [**?**, Theorem 1.5] *proved that if V is a valuation domain with $X^{(1)}(V) = \varnothing$, then $V[\![X_1, \dots ,X_n]\!]_{V-(0)}$ is an n-dimensional (Noetherian) regular local ring. It is shown* [**?**, Lemma 1.2] *that $V[\![X]\!]_{V-(0)} = \bigcup \{ V_P[\![X]\!] \mid P$ is a nonzero prime ideal of $V\} = K[\![X]\!] \cap L$ where L is the quotient field of $V[\![X]\!]$. We remark that while it is possible to have $V[\![X]\!]_{V-(0)} = K[\![X]\!]$ (*[**?**, Example]*), there do exist anti-Archimedean valuation domains that do not satisfy this equality.*

Using a different approach we will generalize Arnold's result to an infinite-dimensional SFT Prüfer domain D. Namely we prove that the ring $D[\![\{X_\alpha\}]\!]_{1_{D-(0)}}$ is a Krull domain for any set of indeterminates $\{X_\alpha\}$ if D is an SFT Prüfer domain. The proof divides into two parts. First, we work on the case where $X^{(1)}(D) = \varnothing$. We show that if D is an anti-Archimedean domain, then $D[\![X_1, \dots, X_n]\!]_{D-(0)}$ is an n-dimensional regular local ring. It then easily follows that if D is anti-Archimedean, then $D[\![\{X_\alpha\}]\!]_{1_{D-(0)}}$ is a UFD. This handles the case for an SFT Prüfer domain D with $X^{(1)}(D) = \varnothing$ since by Proposition **??** *such a domain is anti-Archimedean. Next, given an SFT Prüfer domain D, we decompose D into $D = D_1 \cap D_2$ where D_1 and D_2 are overrings of D with $X^{(1)}(D_1) = \varnothing$ and every maximal ideal of D_2 contains a height-one prime ideal. Although Arnold's result is stated and proved for a finite-dimensional SFT Prüfer domain, a careful reading of the proof shows that it is also valid for an SFT Prüfer domain whose every maximal ideal contains a height-one prime ideal, namely for D_2.*

We first note that $D[\![\{X_\alpha\}]\!]_{i_{D-(0)}}$ is quasilocal if and only if D is anti-Archimedean.

Lemma 3.1. *Let D be an integral domain and $\{X_\alpha\}$ $a nonempty set of indeterminates over D. Then D[\![\{X_\alpha$ $]\!]_{i_{D-(0)}}$ (for $i = 1$, 2, or 3) is quasilocal if and only if D is anti-Archimedean. In this case,*

$M_{D-(0)}$ is the unique maximal ideal of $D[\![\{X_\alpha\}$ $]\!]_{i_{D-(0)}}$, where $M = \{f \in D[\![\{X_\alpha\}$ $]\!]_i \mid f$ has no constant term.

Proof. ($\Rightarrow$) Suppose that $D[\![\{X_\alpha\}$ $]\!]_{i_{D-(0)}}$ is quasilocal. Then $(\{X_\alpha\}$ $)_{D-(0)}$ is contained in the maximal ideal of $D[\![\{X_\alpha$ $]\!]_{i_{D-(0)}}$. Thus for $0 \neq a \in D$, $X_\alpha \overline{a \in (\{X_\alpha\})_{D-(0)}}$ *gives that* $1-$

$X_\alpha \overline{a}$ *is a unit in* $D[\![\{X_\alpha$ $]\!]_{i_{D-(0)}}$. *Hence* $1 + \frac{X_\alpha}{a} +$

$X_\alpha^2 \overline{a^2 + \cdots = (1-}$

$X_\alpha \overline{a^{-1} \in D[\![\{X_\alpha\}}$

$]\!]_{i_{D-(0)}}$. *Hence there exists* $0 \neq r \in D$ *with* $r\,(\frac{1}{}$ $a^n \in D$ *for each* $n \geq 1$, *or equivalently,* $0 \neq r \in \bigcap_{n=1}^{\infty} a^n D$. *($\Leftarrow$) Suppose that D is anti-Archimedean. We show that* $M_{D-(0)}$ *is the unique maximal ideal of* $D[\![\{X_\alpha\}]\!]_{i_{D-(0)}}$. *Let* $f \in D[\![\{X_\alpha\}]\!]_i - M$ *and put* $f = a + g$ *where* $0 \neq a \in D$ *and* $g \in M$. *Let* f^{-1} *be the inverse of* f *in* $D[\frac{1}{a}][\![\{X_\alpha\}]\!]_i$. *Since* $\bigcap_{n=1}^{\infty} a^n D \neq 0$, *there exists* $0 \neq r \in D$ *with* $r\,(\frac{1}{a^n}$ $\in D$ *for each* $n \geq 1$ *and hence* $rf^{-1} \in D[\![\{X_\alpha\}]\!]_i$. *So* $f^{-1} \in D[\![\{X_\alpha$ $]\!]_{i_{D-(0)}}$ *and thus* f *is a unit in* $D[\![\{X_\alpha\}]\!]_{i_{D-(0)}}$.

□

Lemma 3.2. *Let D be an anti-Archimedean integral domain. Let Q be an ideal of $D[\![X_1, \ldots, X_n]\!]$ with $Q \cap D = 0$. Then there exists a nonzero $v \in D$ and $q_1, \ldots, q_l \in Q$ such that*

$$vQ \subseteq (q_1, \ldots, q_l)D[\![X_1, \ldots, X_n]\!].$$

Proof. For brevity of notation, we write $D[\![X_n]\!]$ for $D[\![X_1, \ldots, X_n]\!]$ for each $n \geq 1$ and X for X_n. We will prove the Lemma by induction on the number of variables n. We will deal with the case $n = 1$ later. Suppose the Lemma holds for $n-1$. Let Q be a prime ideal of $D[\![X_n]\!]$ such that $Q \cap D = (0)$. First we show that Q satisfies (**??**). We regard the elements of Q as power series over $D[\![X_{n-1}]\!]$ in X_n and define $\mathcal{I}(Q)$ to be the ideal of $D[\![X_{n-1}]\!]$ generated by the constant terms of Q. If $X \in Q$, then $Q = Q_0 + (X)$ for a prime ideal Q_0 of $D[\![X_{n-1}]\!]$ such that

$Q_0 \cap D = (0)$. In this case, clearly Q satisfies (**??**). Now assume that $X \notin Q$. Let $f^* \in Q$ and $f^* = X^m(f_0(X_1, \ldots, X_{n-1}) + f_1(X_1, \ldots, X_{n-1})X + \cdots + f_i(X_1, \ldots, X_{n-1})X^i + \cdots)$, where $m \geq 0$ and $f_0 \neq 0$. Put $f = \sum_{i=0}^{\infty} f_i X^i$. Since $X \notin Q$, $f \in Q$. By Lemma Lem2.1, $Q \subseteq (X_1, \ldots, X_n)$, so $\mathcal{I}(Q) \cap D = (0)$. Note that $\mathcal{I}(Q) \subseteq D[\![X_{n-1}]\!]$. By induction hypothesis (**??**), there exists $0 \neq v \in D$ and $q_{10}, \ldots, q_{l0} \in \mathcal{I}(Q)$ such that $v(\mathcal{I}(Q)) \subseteq (q_{10}, \ldots, q_{l0})D[\![X_{n-1}]\!]$. Let $q_1, \ldots, q_l \in Q$ be such that $q_i = q_{i0} + \sum_{j=1}^{\infty} q_{ij} X^j$, where $q_{ij} \in D[\![X_{n-1}]\!]$ for $i = 1, \ldots, l$ and $j \geq 1$. Now $v f_0(X_1, \ldots, X_{n-1}) = q_{10}h_{11} + \cdots + q_{l0}h_{1l}$, where $h_{11}, \ldots, h_{1l} \in D[\![X_{n-1}]\!]$. So $f_0 = \frac{1}{v}[q_{10}h_{11} + \ldots + q_{l0}h_{1l}$. Then

$$\phi_1 := f - \frac{1}{v}[q_1 h_{11} + \cdots + q_l h_{1l}] \in D[\tfrac{1}{v}][\![X_n]\!]$$

and

$$\phi_1|_{X_n=0} = f|_{X_n=0} - \frac{1}{v}[q_1 h_{11} + \ldots + q_l h_{1l}{}_{X_n=0}$$

$$= f_0 - \frac{1}{v}[q_{10}h_{11} + \cdots + q_{l0}h_{1l}] = 0.$$

So $X|\phi_1$ in $D[\frac{1}{v}][\![X_n]\!]$. Write $\phi_1 = X^{l_1}\varphi_1$, $l_1 \geq 1$, $\varphi_1 \in D[\frac{1}{v}][\![X_n]\!]$, and $X \nmid \varphi_1$ in $D[\frac{1}{v}][\![X_n]\!]$. Now $f = \frac{1}{v}[q_1 h_{11} + \cdots + q_l h_{1l}] + X^{l_1}\varphi_1$. Since $vX^{l_1}\varphi_1 \in Q$, $v\varphi_1 \in D[\![X_n]\!]$ and $X \notin Q$, we have $v\varphi_1 \in Q$. Applying the same argument to $v\varphi_1$ as to f, we obtain the expression $v\varphi_1 = \frac{1}{v}[q_1 h_{21} + \ldots + q_l h_{2l} + X^{l_2}\varphi_2$, where $l_2 \geq 1$, $h_{21}, \ldots, h_{2l} \in D[\![X_{n-1}]\!]$, $v\varphi_2 \in D[\![X_n]\!]$, and $X \nmid v\varphi_2$ in $D[\![X_n]\!]$. Plugging this into f, we get

$$f = 1_{v[q_1 h_{11} + \cdots + q_l h_{1l}] + X^{l_1}\varphi}$$

1

$=1\frac{}{v[q_1h_{11}+\cdots+q_lh_{1l}]+X^{l_1}[}$
$1\frac{}{v^2[q_1h_{21}+\cdots+q_lh_{2l}]+1}$
$\frac{}{vX^{l_1+l_2}\varphi_2.}$

Proceeding this way, we get, for all $j \geq 1$,

$f=\sum_{i=1}^{j} \frac{1}{v^i} [q_1h_{i1} + \cdots + q_lh_{il}]$
$X^{l_1+\cdots+l_{i-1}} + \frac{1}{v^{j-1}} X^{l_1+\cdots+l_j}\varphi_j,$

where $h_{ik} \in D[\![X_{n-1}]\!]$ for all i, k, $l_i \geq 1$ for all i=1, ... ,$j, and v\varphi_j \in D[\![X_n]\!]$. When $i = 1$, the sum $l_1 + \cdots + l_{i-1}$ is taken to be zero. Letting $j \to \infty$, we obtain $f = \sum_{i=1}^{\infty} \frac{1}{v^i} [q_1h_{i1} + \ldots$ $+q_lh_{il}X^{l_1+\cdots+l_{i-1}}$. Since $\bigcap_{i=1}^{\infty}$ $(v^i) \neq 0$, there exists a nonzero $w \in \bigcap_{i=1}^{\infty}(v^i)$. Since $wD[\frac{1}{v}] \subseteq D$, the element $\frac{w}{v^i} =: w_i \in D for i \geq 1$. Now $wf = q_1(\sum_{i=1}^{\infty} w_ih_{i1}X^{l_1+\cdots}$ $+l_{i-1}) + \cdots + q_l(\sum_{i=1}^{\infty} w_ih_{il}X^{l_1+\cdots}$ $+l_{i-1}) \in (q_1, \ldots, q_l)D[\![X_n]\!]$. Since $0 \neq w \in D$, $q_1, \ldots, q_l \in Q$, and w does not depend on the particular element $f of Q, (??) is satisfied for all prime ideals.$

Let $\mathcal{S} = \{J \mid J$ is an ideal of $D[\![X_n]\!]$, $J \cap D = (0)$, and J does not satisfy (??)$\}$. We claim that S=$\varnothing$. Suppose $\mathcal{S} \neq \varnothing$. Let $\{J_\alpha\}$ be a chain in $\mathcal{S}$ and J=$\bigcup_\alpha J_\alpha$. Then J does not satisfy (??). For otherwise, there exist $q_1, \ldots, q_l \in J$ and $0 \neq v \in D$ such that $vJ \subseteq (q_1, \ldots, q_l)$. Choose J_α so that $(q_1, \ldots, q_l) \subseteq J_\alpha$. Then $vJ_\alpha \subseteq$ $(q_1, \ldots, q_l)$, a contradiction. Thus $J \in \mathcal{S}$. By Zorn's Lemma, there exists a maximal element of $\mathcal{S}$, say Q. We may assume that $Q^{ec} := Q_{D-(0)} \cap D[\![X_n]\!] = Q$. For otherwise $Q \subsetneq Q^{ec}$ and there exist $q_1, \ldots, q_l \in Q^{ec}$ and $0 \neq$ $v \in D$ such that $vQ^{ec} \subseteq (q_1, \ldots, q_l)$. Choose $0 \neq$ $v' \in D$ such that $v'q_1, \ldots, v'q_l \in Q$. Then $v'vQ \subseteq v'(q_1, \ldots, q_l) = (v'$ $q_1, \ldots, v'q_l)$, a contradiction. We claim that Q is a prime ideal. Suppose not and let $f \in D[\![X_n]\!] \setminus Q$, $g \in$ $D[\![X_n]\!] \setminus Q$, and $fg \in Q$. Either $(Q, f) \cap D = (0)$ or $(Q,g) \cap D = (0)$. For otherwise $(0) \neq [(Q, f)(Q, g)] \cap$ $D \subseteq Q \cap D$. Say $(Q, f) \cap D = (0)$. By the maximality of Q,

*$(Q,f)\notin \mathcal{S}$. So (Q,f) satisfies (**??**). Let $0\neq v\in D$ and $a_1,\dots,a_l\in(Q,f)$ such that $v(Q,f)\subseteq (a_1,\dots,a_l)D[\![X_n]\!]$. So $v(Q,f)\subseteq(a_1,\dots,a_l)D[\![X_n]\!]$. Put $a_1=q_1+t_1f,\dots,a_l=q_l+t_lf$, where $q_1,\dots,q_l\in Q$ and $t_1,\dots,t_l\in D[\![X_n]\!]$. Since $v(Q,f)\subseteq(q_1,\dots,q_l,f)D[\![X_n]\!]$ and $q_1,\dots,q_l\in Q$, it is straightforward that $v(Q,f)\subseteq(q_1,\dots,q_l)+f(Q:f)$, where $(Q:f)$ is the ideal $\{h\mid h\in D[\![X_n]\!]$ and $hf\in Q$. We claim that $(Q:f)\cap D=(0)$. Suppose not and let $0\neq w_1\in(Q:f)\cap D$. Now $w_1f\in Q$. Then $f\in Q_{D-(0)}\cap D[\![X_n]\!]=Q$, contrary to $f\notin Q$. By the maximality of Q, $(Q:f)$ satisfies (**??**). Let $0\neq w\in D$, and $h_1,\dots,h_m\in(Q:f)$ be such that $w(Q:f)\subseteq(h_1,\dots,h_m)D[\![X_n]\!]$. Now*

$$vwQ\subseteq vw(Q,f)=w\left[v(Q,f)\right]\subseteq w\left[(q_1,\dots,q_l)+f(Q:f)\right.$$

$$\subseteq w(q_1,\dots,q_l)+f\left[w(Q:f)\right]\subseteq w(q_1,\dots,q_l)+f(h_1,\dots,h_m)D[\![X_n]\!]$$

$$\subseteq(q_1,\dots,q_l,fh_1,\dots,fh_m)D[\![X_n]\!].$$

*Note that $fh_1,\dots,fh_m\in Q$. This contradicts that $Q\notin S. Hence Q is a prime ideal, which contradicts the already$ established fact that all the prime ideals of $D[\![X_n]\!]$ satisfy (eq.Star). Therefore $\mathcal{S}=\varnothing$ and we conclude that every ideal of $D[\![X_n]\!]$ satisfies (**??**).*

*Now we prove the case $n=1$. Every element $f\in D[\![X]\!]$ is of the form $f=X^m(a_0+a_1X+\cdots)$, $a_0\neq0$. Since $(X)_{D-(0)}$ is the maximal ideal of $D[\![X]\!]_{D-(0)}$, $a_0+a_1X+\cdots$ is a unit of $D[\![X]\!]_{D-(0)}$. So the set of nonzero ideals of $D[\![X]\!]_{D-(0)}$ is precisely $\left\{X^iD[\![X]\!]_{D-(0)}\mid i=1,2,\dots\right\}$. Thus $D[\![X]\!]_{D-(0)}$ is a Noetherian ring and $(X)_{D-(0)}$ is the unique nonzero prime ideal of $D[\![X]\!]_{D-(0)}$. So $Q=(X)$ is the unique prime ideal of $D[\![X]\!]$ such that $Q\cap D=(0)$. Clearly Q satisfies (eq.Star) and hence the conclusion that every ideal J of $D[\![X]\!]$ such that $J\cap D=(0)$ satisfies (**??**) follows from the earlier argument.*

□

Theorem 3.3. *Let D be an anti-Archimedean integral domain. Then $D[\![X_1,\dots,X_n]\!]_{D-(0)}$ is an n-dimensional regular local ring.*

Proof. By Lemmas **??** and **??**, $D[\![X_1,\dots,X_n]\!]_{D-(0)}$

$isaNoetherianlocalringwiththemaximalideal$(X$_1, \ldots$
,X$_n)_{D-(0)}$. Since ht$(X_1, \ldots, X_n)_{D-(0)} \geq n$, it
follows that ht$(X_1, \ldots, X_n)_{D-(0)} = n$. Hence
D$[\![$X$_1, \ldots, X_n]\!]_{D-(0)}$ is a regular ring
AM.

□

Corollary 3.4. *Let D be an anti-Archimedean domain. Then for any set of indeterminates $\{X_\alpha\}$, $D[\![\{$
$X_\alpha]\!]_{1_{D-(0)}}$ is a UFD.*

Proof. Let Q be a nonzero prime ideal of $D[\![\{X_\alpha\}$
$]\!]_{1_{D-(0)}}$ and let $0 \neq f \in Q$. Since f involves only finitely many indeterminates, say, $X_1, \ldots, X_n$, we have $0 \neq f \in Q'$
=Q$\cap D[\![X_1, \ldots, X_n]\!]_{D-(0)}$. Since $D[\![X_1, \ldots$
,X$_n]\!]_{D-(0)}$ is a regular local ring and hence a UFD, there is a prime element $q \in Q'$. It suffices to check that q remains prime as an element of $D[\![\{X_\alpha\}]\!]_{1_{D-(0)}}$
. For then Q contains a nonzero principal prime and thus by [**?**, Theorem 5] $D[\![\{X_\alpha\}]\!]_{1_{D-(0)}}$ is
a UFD. Suppose that $q | f_1 f_2$ in $D[\![\{X_\alpha\}$
$]\!]_{1_{D-(0)}}$. We can enlarge the set $\{X_1, \ldots$
,X$_n$ to a subset $\{X_1, \ldots, X_m\}$ of
$\{X_\alpha\}$ so that $q, f_1, f_2 \in D[\![X_1, \ldots$
,X$_m]\!]_{D-(0)}$ and $q | f_1 f_2$ in $D[\![X_1, \ldots$
,X$_m]\!]_{D-(0)}$. It suffices to show that q is prime in
D$[\![$X$_1, \ldots, X_m]\!]_{D-(0)}$. And since $D[\![X_1, \ldots$
,X$_m]\!]_{D-(0)}$ is a UFD, we only need show that q is irreducible in
$D[\![X_1, \ldots, X_m]\!]_{D-(0)}$. Suppose that $q = h_1 h_2$ in
D$[\![$X$_1, \ldots, X_m]\!]_{D-(0)}$. Then $q = q(X_1, \ldots, X_n) = q($
X$_1, \ldots, X_n, X_{n+1}, \ldots, X_m) = q(X_1, \ldots, X_n, 0, \ldots, 0)$
=h$_1(X_1, \ldots, X_n, 0, \ldots, 0) h_2(X_1, \ldots, X_n, 0, \ldots, 0)$
$.Sosay$h$_1(X_1, \ldots, X_n, 0, \ldots, 0)$ is a unit in
D$[\![$X$_1, \ldots, X_n]\!]_{D-(0)}$. Thus $h_1(0, \ldots, 0, 0, \ldots, 0)$
$\neq 0$ in D and hence h_1 is a unit in $D[\![X_1, \ldots$
,X$_m]\!]_{D-(0)}$.

□

Lemma 3.5. *Let D be an SFT Prüfer domain. Let $\varnothing \neq$
$\Lambda \subseteq$ Spec$(D) - \{0\}$ and put
$D_0 = \bigcap_{Q \in \Lambda} D_Q$.
4)(*
1) Suppose that no $P \in$
Λ contains a height-one prime ideal. Then no prime ideal of D_0 contains a height-one prime ideal.
)(2
) Suppose that each $P \in \Lambda$ contains a height-one
prime ideal. Then each nonzero prime ideal of D_0 contains a height-one prime ideal.

Proof. Let M be a nonzero prime ideal of D_0. By [**?**, Theorem 26.1]

M=PD$_0$ for some prime ideal P (namely $P = M \cap D$) of D and D$_P \supseteq D_0 = \bigcap_{Q\in\Lambda} D_Q$. But by [?, Exercise 16, page 332], $D_P \supseteq \bigcap_{Q\in\Lambda}$ D$_Q$ $\Leftrightarrow$ each finitely generated ideal I contained in P is contained in some $Q \in \Lambda$ (depending on I). Since D is an SFT-ring, $P = \sqrt{(a_1, \ldots, a_n)}$ for some $a_1, \ldots, a_n \in P$. Then $(a_1, \ldots, a_n) \subseteq Q'$ for some $Q' \in$ Λ and hence $P = \sqrt{(a_1, \ldots, a_n)} \subseteq Q'$. (**??**) Let M be a nonzero prime ideal of D_0 with the notation as above. If M contains a height-one prime ideal P_0, then P$_0 \cap D \subseteq M \cap D = P \subseteq Q' \in \Lambda$ and P$_0 \cap D$ is a height-one prime ideal contained in Q', a contradiction. (**??**) Again, let M be a nonzero prime ideal of D$_0$ with the notation as above. Now $M = PD_0$ where $P \subseteq$ Q$' \in \Lambda$ and by hypothesis Q' contains a height-one prime ideal P_0. Since the prime ideals contained in Q$'$ are totally ordered, $P_0 \subseteq P$. Then P$_0D_0 \subseteq PD_0 = M$ and P_0D_0 is a height-one prime ideal of D_0.

□

Theorem 3.6. *Let D be an SFT Prüfer domain. Then $D = D_1 \cap D_2$ where D_1 and D_2 are overrings of D (necessarily SFT Prüfer domains) such that $X^{(1)}(D_1) = \varnothing$ and every nonzero prime ideal of D_2 contains a height-one prime ideal.*

Proof. Let Λ_1 be the set of maximal ideals of D that contain no height-one prime ideals and let Λ_2 be the set of maximal ideals of D that do contain a height-one prime ideal. Put $D_i = \bigcap_{Q\in}$ $\Lambda_i D_Q$. Now Max$(D) = \Lambda_1 \dot\cup \Lambda_2$, so $D = D_1 \cap D_2$. Moreover, by [?, Proposition 2.4] each D$_i$ is an SFT Prüfer domain. By Lemma **??**, $X^{(1)}($ D$_1) = \varnothing$ and every nonzero prime ideal of D_2 contains a height-one prime ideal.

□

Theorem 3.7. *Let D be an SFT Prüfer domain. Then $D[\![\{$ $X_\alpha]\!]_{1_{D-(0)}}$ is a Krull domain for any set of indeterminates $\{X_\alpha\}$.*

Proof. Write $D = D_1 \cap D_2$ as in Theorem **??**. Since D_1 is an SFT Prüfer domain with $X^{(1)}(D_1) = \varnothing$, by Proposition Pro1.3, D_1 is anti-Archimedean and hence by Corollary Cor2.4 $D_1[\![\{X_\alpha\}]\!]_{1_{D_1-(0)}}$ is a UFD. Now D_2 is an SFT Prüfer domain such that every maximal ideal of D_2 contains a height-one prime ideal. Now [?, Lemma 3.5] shows that $D_2[\![X_1, \ldots, X_n]\!]_{D_2-(0)}$ is a Noetherian Krull domain in the case where $\dim D_2 < \infty$ (for by the second paragraph of page 304 of [?], J_n is a Krull domain and hence so is J$_{n_{D_2-(0)}} = D_2[\![X_1, \ldots, X_n]\!]_{D_2-(0)}$). But a careful reading of the proof of [?, Lemma 3.5], and the material in Section 3 preceding it, shows that we only need D_2 to be an SFT Prüfer domain in which every maximal ideal contains a height-one prime ideal. Thus $D_2[\![X_1, \ldots, X_n]\!]_{D_2-(0)}$ is a Noetherian Krull domain. We next observe that $D_2[\![\{X_\alpha\}]\!]_{1_{D_2-(0)}}$ is actually a Krull domain for any set $\{X_\alpha\}_{\alpha\in\Lambda}$ of indeterminates. We extend Arnold's method to an arbitrary set of indeterminates as follows. The integral domain J defined in [?, page 902] is a Dedekind domain (even when $\dim D_2 = \infty$) and thus $J[\![\{X_\alpha\}]\!]_1$ is a Krull domain. For $\Lambda' \subseteq \Lambda$, let $K_{\Lambda'}$ be the quotient field of $D_{\Lambda'} = D_2[\![\{X_\alpha \mid \alpha \in \Lambda'\}]\!]_1$ and $J_{\Lambda'} = J[\![\{X_\alpha \mid \alpha \in \Lambda'\}]\!]_1 \cap K_{\Lambda'}$. Thus by Arnold's work (extended to the case $\dim D_2 = \infty$) we have $(J_F)_{D_2-(0)} = D_2[\![\{X_\alpha \mid \alpha \in F\}]\!]_{D_2-(0)}$ for each finite subset $\mathcal{F} \subseteq \Lambda$. Now

$$D_2[\![\{X_\alpha\}]\!]_{1_{D_2-(0)}} = \bigcup_{F\subseteq\Lambda,\ |\mathcal{F}|<\infty} D_2[\![\{X_\alpha \mid \alpha \in \mathcal{F}\}]\!]_{D_2-(0)} = \bigcup_{\mathcal{F}\subseteq\Lambda,\ |\mathcal{F}|<\infty} (J_{\mathcal{F}})_{D_2-(0)} = \Big(\bigcup_{\mathcal{F}\subseteq\Lambda,\ |F|<\infty} J_{\mathcal{F}}\Big)_{D_2-(0)} = (J_\Lambda)_{D_2-(0)}$$

which is a Krull domain since J_Λ is a Krull domain. Note that $D_i[\![\{X_\alpha\}]\!]_{1_{D-(0)}} = D_i[\![\{X_\alpha\}]\!]_{1_{D_i-(0)}}$. Hence $D[\![\{X_\alpha\}]\!]_{1_{D-(0)}} = D_1[\![\{X_\alpha\}]\!]_{1_{D-(0)}} \cap D_2[\![\{X_\alpha\}]\!]_{1_{D-(0)}} = D_1[\![\{X_\alpha\}]\!]_{1_{D_1-(0)}} \cap D_2[\![\{X_\alpha\}]\!]_{1_{D_2-(0)}}$ is the intersection of two Krull domains and hence is a Krull domain. □

References

[1] D.D. Anderson, *The Krull intersection theorem*, Pacific J. Math. **57** (1975), 11–14.

[2] D.D. Anderson, D.F. Anderson, and R. Markanda, rings $R(X)$ and $R\langle X\rangle$ *The rings $R(X)$ and $R\langle X\rangle$, J. Algebra **95** (1985), 96–115.*

[3] D.F. Anderson and D.E. Dobbs,
*same prime ideals*Pairs of rings with the
same prime ideals, Canad. J. Math. **32** (1980), 362–384.
[4] J.T. Arnold, *Krull dimension in power series rings*,
Trans. Amer. Math. Soc. **177** (1973), 299–304.
[5] J.T. Arnold,
Power series rings over Prüfer domains
, Pacific J. Math. 44 (1973), 1–11.
[6] J.T. Arnold,
*rings*Power series rings over discrete valuation
rings, Pacific J. Math. **93** (1981), 31–33.
[7] J.T. Arnold,
dimension*Power series rings with finite Krull*
*dimension, Indiana Univ. Math. J. **31** (1982), 897–911.*
[8] M.F. Atiyah and I.G. Macdonald,
*ommutative Algebra*Introduction to C
ommutative Algebra, Addison-Wesley Publishing Co., Reading, Mass., 1969.
[9] R.A. Beauregard and D.E. Dobbs,
rchimedean integral domains*On a class of A*
*rchimedean integral domains, Canad. J. Math. **28** (1976), 365–375.*
[10] A.M. de S. Doering and Y. Lequain,
*ideals in polynomial rings*Chains of prime
ideals in polynomial rings, J. Algebra **78** (1982), 163–180.
[11] R. Gilmer,
⟦X⟧$Anoteonthequotientfieldofthedomain$D
⟦X⟧$, Proc. Amer. Math. Soc. \mathbf{18} (1967), 1138--1140.$
[12] R. Gilmer, *Multiplicative Ideal Theory*, Queen's
Papers in Pure and Applied Mathematics, vol. 90, Queen's University,
Kingston, Ontario, 1992.
[13] B.G. Kang and M.H. Park,
series ring over a valuation domain*A localization of a power*
series ring over a valuation domain, preprint.
[14] I. Kaplansky, Commutative Rings*, revised ed., Polygonal*
Publishing House, Washington, N.J., 1994.
[15] T. Nakayama,
completely integrally closed integrity domains, I, IIOn Krull's conjecture concerning
completely integrally closed integrity domains, I, II, Proc. Imp.
Acad. Tokyo **18** (1942), 185–187, 233–236; III, Proc. J
apan Acad. **22** (1946), 249–250.
[16] J. Ohm,
in $D[\![x]\!]$*Some counterexamples related to integral closure*
in $D[\![x]\!]$*, Trans. Amer. Math. Soc. **122** (1966), 321–333.*

The University of Iowa,
Iowa City, IA 52242 U.S.A.
@
uiowa.edu
Pohang Institute of Science and Technology,
Pohang, 790-784 Korea
@
posmath.postech.ac.kr