

Discrete Math. Graduation Exam, Spring 2022

1. The *XOR*-gate is a function $\oplus : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, written as $\oplus(x, y) = x \oplus y$, defined as

$$x \oplus y := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Suppose you want to construct the *XOR*-gate by using the *AND*, *OR* and *NOT* gates only. Show that the *XOR*-gate can be constructed by using less than five gates.

2. Let G be a bipartite graph with bipartition V and W . A *matching from V to W* is a set E of edges with no vertices in common. A *complete matching from V to W* is a matching such that if $v \in V$, then there is $(v, w) \in E$ for some W . Explain how to obtain a complete matching, if exists, by using the Ford-Fulkerson algorithm.
3. Find a solution of the linear non-homogeneous recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} + (n+1)3^n, \quad n \geq 2$$

where $a_0 = 0$ and $a_1 = 0$.