Qualifying Examination - Analysis - 01/2022.

1. (*15 points*)
(a) (5 points) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

(b) (5 points) Compute

$$\lim_{n\to\infty} \frac{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}}{\sqrt{n}}.$$

(c) (5 points) Prove that the set of all polynomials in $C([a,b], \mathbf{R})$ is not open.

2. (*10 points*) Assume that 2π-periodic continuous real-valued function $f : \mathbf{R} \to \mathbf{R}$ also satisfies

$$f(x) = f(x+1)$$

for all $x \in \mathbf{R}$. Show that f is constant.

3. (*10 points*) Let A be a compact set in $\mathbb{R}^n$ and B be an open set which contains A, i.e., $A \subset B$.

(a) (5 points) Show that there exists $\varepsilon > 0$ such that an open ball $B_\varepsilon(x) \subset B$ for any $x \in A$.

(b) (5 points) Does the statement (a) hold even if A is just closed? Prove or provide a counter example.

4. (*15 points*) Consider $f_n(x) = \sin(nx)$ for $-\pi \le x \le \pi$ and $n = 1, 2, 3, \cdots$.

(a) (5 points) Show that $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^2[-\pi,\pi]$ where $L^2[-\pi,\pi]$ is square Lebesgue integrable function space.

(b) (5 points) Show that $\{f_n\}_{n=1}^{\infty}$ is closed in $L^2[-\pi,\pi]$.

(c) (5 points) Is $\{f_n\}_{n=1}^{\infty}$ compact in $L^2[-\pi,\pi]$? Prove or disprove.

5. (*10 points*) Find two solutions of the initial value problem

$$y' = \sqrt{y^2 - 1}, \quad y(0) = 1.$$

Explain why there does not exist a unique solution.

6. (*10 points*) Consider the initial value problem

$$\frac{d\mathbf{y}}{dt} = \Omega \mathbf{y}, \qquad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0$$

where $\Omega = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ and $\mathbf{y}_0 = (0,0,1)^T$. Find $\mathbf{y}(\frac{\pi}{2})$.

7. (*10 points*) Evaluate

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta.$$

8. (*10 points*) Let f be analytic in the whole complex plane. Assume that f satisfies

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq r^{\sqrt{e}},$$

for all $r > 0$. Then prove that $f \equiv 0$.

9. (*10 points*) Suppose f and g are entire functions with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that $f(z) = cg(z)$ for some constant c.