Algebra Qualifying Exam

January 26, 2022

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1. (5 pts each) Let $\mathbb{F}_q$ be the finite field of q elements.
   (a) Define a suitable group action of $\mathrm{GL}_3(\mathbb{F}_q)$ on $\mathrm{M}_3(\mathbb{F}_q)$ such that each orbit is a conjugacy class of a matrix in $\mathrm{M}_3(\mathbb{F}_q)$. Here, $\mathrm{M}_3(\mathbb{F}_q)$ is the set of 3×3 matrices with entries in $\mathbb{F}_q$ and $\mathrm{GL}_3(\mathbb{F}_q)$ is the group of invertible matrices in $\mathrm{M}_3(\mathbb{F}_q)$.
   (b) Find the number of orbits in (a) consisting of matrices whose trace and determinant are zero.
   (c) Let $q = 3$. Find the Jordan canonical form of the matrix
   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$
   (d) Let $q = 3$. Compute the number of matrices which are conjugate to the matrix
   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$
   You can use the formula $\#\mathrm{GL}_3(\mathbb{F}_q) = (q^3 - 1)(q^3 - q)(q^3 - q^2)$, without proving it.

2. (10pts each) Suppose that a group homomorphism $\varphi : G \to H$ is surjective, where G is a finite group.
   (a) If P is a Sylow p-subgroup of G, then prove that $\varphi(P)$ is also a Sylow p-subgroup of H.
   (b) If P_1 and P_2 are Sylow p-subgroups of G with $\varphi(P_1) = \varphi(P_2)$, then prove that P_1 and P_2 are conjugate by an element of $\ker \varphi$.

3. (10pts each)
   (a) Let $f(x) = 5x^5 + 6x^2 + 2$. Show $M = (5, f(x))$ is a maximal ideal in $\mathbb{Z}[x]$.
   (b) Let $\alpha = x + M$ in $\mathbb{Z}[x]/M$. Find the inverse of $2\alpha^3 + 1$.

4. (10pts each)
   (a) Show $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.
   (b) Using the fact that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain, find all the integer points of $y^2 = x^3 - 2$.

5. (20 pts) Let $\mathbb{Q}$ be a field of rational numbers. Suppose

$$\mathbb{Q}(3^{\frac{1}{3}}) \leq E \leq R \leq \overline{\mathbb{Q}},$$

where E is a finite Galios extension of $\mathbb{Q}(3^{\frac{1}{3}})$ and $R = \mathbb{Q}(3^{\frac{1}{3}}, a, b)$, where $a^n \in \mathbb{Q}(3^{\frac{1}{3}})$ and $b^m \in \mathbb{Q}(3^{\frac{1}{3}}, a)$ for positive integers m, n.

   (a) (15 pts) Is it possible to construct a following field K? Justify your answer.
      i. $R \leq K \leq \overline{\mathbb{Q}}$
      ii. K is a finite Galois extension of $\mathbb{Q}(3^{\frac{1}{3}})$
      iii. K is an extension of $\mathbb{Q}(3^{\frac{1}{3}})$ by radicals
      iv. $Gal(K/\mathbb{Q}(3^{\frac{1}{3}}))$ is solvable
   (b) (5 pts) Can we conclude that $Gal(E/\mathbb{Q}(3^{\frac{1}{3}}))$ is solvable? Justify your answer.