Algebra Qualifying Exam

1. (16pts) A group G is *perfect* if the commutator subgroup $[G, G]$ is equal to G.
   (a) Show that $G = \langle x, y \mid x^5 = (xy)^2 = y^3 \rangle$ is perfect.
   (b) Prove or disprove: every quotient of a perfect group is perfect.
   (c) Prove or disprove: every subgroup of a perfect group is perfect.

2. (16pts) A group G is *Hopfian* if G/H is not isomorphic to G for any nontrivial normal subgroup H.
   (a) Show that not all abelian groups are Hopfian.
   (b) Show that every finitely generated abelian group is Hopfian.

3. (17pts) Let $\mathfrak{P}$ be a prime ideal of $\mathbb{Z}[x]$.
   (a) Show that $\mathfrak{P} \cap \mathbb{Z}$ is either (0) or (p) for a certain prime number p.
   (b) If $\mathfrak{P} \cap \mathbb{Z} = (p)$, then show that either $\mathfrak{P} = (p)$ or $\mathfrak{P} = (p, f)$, where $f \in \mathbb{Z}[x]$ reduces to an irreducible polynomial in $\mathbb{F}_p[x]$ modulo p.

4. (17pts) Let $\mathbb{F}_q$ be the finite field of q elements.
   (a) For an arbitrary prime integer p, prove that the number of monic irreducible polynomials of degree p in $\mathbb{F}_q[x]$ is
   $$\frac{q^p - q}{p}.$$
   Hint: Consider the splitting field of such polynomial.
   (b) Find the number of conjugacy classes in the group $\mathrm{GL}_3(\mathbb{F}_q)$ which does not have eigenvalues in $\mathbb{F}_q$. You must justify your answer.

5. (17pts) We use the notation $A \lhd B$ if B is a normal extension (finite separable, splitting) of a field A. Let $\mathbb{B}$ be the splitting field of $x^6 - 2^{\frac{1}{5}}$ over $\mathbb{Q}(2^{\frac{1}{5}}, \xi)$, where $\xi = e^{\frac{2\pi i}{6}}$.
   (a) Show that $\mathbb{Q}(2^{\frac{1}{5}}, \xi) \lhd \mathbb{B}$.
   (b) Show that $\mathrm{Gal}(\mathbb{B}/\mathbb{Q}(2^{\frac{1}{5}}, \xi))$ is abelian.

6. (17pts) We keep the notation $A \lhd B$ if B is a normal extension of A. Let $\mathbb{F} = \mathbb{Q} \lhd \mathbb{F}' = \mathbb{Q}(w, 5^{\frac{1}{3}}) < \mathbb{F}'(\gamma)$, where $\gamma = (1 + 5^{\frac{1}{3}})^{\frac{1}{4}}, w = e^{\frac{2\pi i}{3}}$. Construct $\mathbb{F}''$ such that
   (a) $\mathbb{F} \lhd \mathbb{F}' < \mathbb{F}'(\gamma) \lhd \mathbb{F}''$,
   (b) $\mathbb{F} \lhd \mathbb{F}''$,
   (c) $\mathbb{F}''$ is a radical extension of $\mathbb{F}'$, and
   (d) $\mathrm{Gal}(\mathbb{F}''/\mathbb{F}')$ is solvable.

   Can we conclude that $\mathrm{Gal}(\mathbb{F}''/\mathbb{F})$ is solvable?