Qualifying Examination - Analysis - 07/2021.

1. (10pts) Consider the series $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+4}$.

   (a) Does the series converge absolutely?

   (b) Does the series converge?

2. (10pts) Evaluate

$$\lim_{n\to\infty}\int_0^1 x^n f(x)dx$$

where f is a continuous real valued function on $[0,1]$. (Show by explicit estimate instead of using Dominated convergence theorem or Monotone convergence theorem)

3. (10pts) Consider a continuous function $f : E \subset \mathbb{R} \to \mathbb{R}$ where E is a closed Lebesgue measurable set.

   (a) Prove that $\lim_{p\to\infty}\left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}} = \sup_{x\in E}|f(x)|$ if $|E| < \infty$.

   (b) Does the equality of (a) hold for $|E| = \infty$ case? Prove or give a counter example.

4. (15pts) Let the functions $f_n : [a,b] \to \mathbb{R},\ n \in \mathbb{N}$ be uniformly (in $n \in \mathbb{N}$) bounded continuous functions. Let

$$F_n(x) := \int_a^x f_n(t)dt, \quad a \le x \le b.$$

Prove that $\{F_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence in $\mathcal{C}([a,b],\mathbb{R})$.

5. (10pts) Let f be an entire function such that $|f(z)| > 1$ whenever $|z| > 1$. Show that $f(z)$ is a polynomial.

6. (10pts) Use contour integration to evaluate

$$\int_0^{\infty}\frac{\cos x}{(1+x^2)^2}dx.$$

7. (10pts) Let f be an analytic function on $\mathbb{C}$ that takes value in the upper half plane $\mathbb{U} := \{z : \operatorname{Im} z > 0\}$. Show that f is constant.

8. (15pts) Consider the following initial value problem

$$y'' - 6y' + 8y = 0, \quad y(0) = 5 \quad y'(0) = 8.$$

   (a) (8pts) Find the solution of the initial value problem above.
   (b) (7pts) Determine the maximum value of the solution.

9. (10pts) Find the solution of the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{pmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$