Algebra Qualifying Exam

January 2021

1. (12pts) Describe the kernel of the following ring homomorphisms and prove it. You must prove why the ideal you describe is the kernel in detail.

   (a) $\varphi : \mathbb{R}[x, y] \to \mathbb{R}$ defined by $\varphi(f(x, y)) = f(0, 0)$.

   (b) $\varphi : \mathbb{R}[x] \to \mathbb{C}$ defined by $\varphi(f(x)) = f(1 - i)$. (Hint: $\mathbb{R}[x]$ is an Euclidean domain.)

2. (10pts) Show that $x^{100} + 81x + 213$ is irreducible in $\mathbb{Q}[x]$. You must use Gauss's lemma and Eisenstein's criterion.

3. (12pts) Let p be a prime number. It is known that $|\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})| = p^4 - p^3 - p^2 + p$. (You do not need to prove this.)

   (a) Find $|\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})|$. Hint: Construct a surjective homomorphism from $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ whose kernel is $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. You must show that your map is surjective and a homomorphism.

   (b) Find an element of order p in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$.

4. (14pts) Let $\mathbb{Q}$ be the field of rational numbers and let K be the splitting field of $x^{10} - 1$.

   (a) Find the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, in terms of the fundamental theorem of finitely generated abelian groups.

   (b) Find the number of subfields of K which are a field extension of $\mathbb{Q}$ of degree 2?

5. (14pts) Prove the following.

   (a) The ring $\mathbb{Q}[t, t^{-1}] = \{\sum_{k\in\mathbb{Z}} a_k t^k \mid a_k \in \mathbb{Q},\ a_k = 0 \text{ for all } k \text{ but finitely many}\}$ of laurent polynomials over $\mathbb{Q}$ is a principal ideal domain.

   (b) A finitely generated module M over $\mathbb{Q}[t, t^{-1}]$ is torsion if and only if the dimension of M over $\mathbb{Q}$ is finite.

6. (10pts) Let G be a free group of rank two, generated by $x, y \in G$. Find a normal subgroup H of G and a normal subgroup K of H such that K has finite index in G and K is not normal in G.

7. (14pts)

   (a) Consider the extension $\mathbb{R}(\alpha)$, where α is a zero of the polynomial $m(x) = x^2 + x + 1 \in \mathbb{R}[x]$. Write the element $u = \frac{3\alpha^2+2}{\alpha+4} \in \mathbb{R}(\alpha)$ in the form $p(\alpha)$ with $p(x) \in \mathbb{R}[x]$, $\deg(p) < 2$.

   (b) Let $K(\alpha)$ be a simple extension of a field K, where α has a minimal polynomial $m(x) \in K[x]$. Show that any element of $K(\alpha)$ has a unique expression in the form $p(\alpha)$, where $p(x) \in K[x]$ and $\deg(p) < \deg(m)$.

8. (14pts) Prove the following.

   (a) Let X be a set and G be a group acting transitively on X. Show that there is a bijective map from X onto G/G_x, where $G_x := \{h \in G \mid h \cdot x = x\}$ is the stabilizer of a point $x \in X$.

   (b) Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $SO(2) = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in SL_2(\mathbb{R}) \mid a^2 + b^2 = 1\}$. Using (a) and the association $z \mapsto \frac{az+b}{cz+d}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$, construct a bijection between H and $SL_2(\mathbb{R})/SO(2)$,