Qualifying Examination - Analysis - 01/2021.

1. (10 pts) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

2. Consider a sequence of real-valued functions

$$f_n(x) = \frac{x}{1+nx^2}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(1) (10pts) Show that $f_n \to f$ uniformly for some f.

(2) (5pts) Does $f'(x) = \lim_{n\to\infty} f_n'(x)$ hold for all $x \in \mathbb{R}$?

3. (10pts) Assume that 2π-periodic continuous real-valued function $f : \mathbf{R} \to \mathbf{R}$ also satisfies

$$f(x) = f(x+1)$$

for all $x \in \mathbf{R}$. Show that f is constant. (Hint : Consider the Fourier coefficient $\hat{f}(k) := \int_0^{2\pi} f(x)e^{-ikx}dx$.)

4. (10pts) Prove that

$$\lim_{n\to\infty} \frac{1}{n}(n!)^{\frac{1}{n}} = e^{-1}$$

by using the definition of Riemann integral $\int_0^1 \log x dx$.

5. (10pts) Find all analytic functions f on $\mathbb{C}\backslash\{0\}$ such that $|f(z)| \le |z|^{-1/2} + |z|^{1/2}$ for $0 < |z| < \infty$.

6. (10pts) Using residues, evaluate the integral

$$\int_0^{\infty} \frac{1}{\sqrt{x}(x^2+1)} dx.$$

7. (10pts) Find a conformal mapping which maps the domain D onto the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ where $D := \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0\}$.

8. Consider the following initial value problem

$$y'' + 3y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = 0$$

(1) (8pts) Find the solution of the initial value problem above.
(2) (7pts) Find $\sup_{t\ge 0} y(t)$ and $\inf_{t\ge 0} y(t)$.

9. (10pts) Suppose that a three-dimensional vector is rotating at a constant rate of 1 rad/sec about the unit axis $\hat{\mathbf{w}} = (w_1, w_2, w_3)^T$ with $\|\hat{\mathbf{w}}\| = 1$. Let $\mathbf{p}(t)$ denote the position of the vector at time t. The velocity of $\mathbf{p}(t)$ is then given by $\frac{d\mathbf{p}}{dt} = \hat{\mathbf{w}} \times \mathbf{p}$. From the fact that $\hat{\mathbf{w}} \times \mathbf{p} = \Omega \mathbf{p}$ where $\Omega = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$, we have the differential equation for $\mathbf{p}(t)$:

$$\frac{d\mathbf{p}}{dt} = \Omega \mathbf{p}, \qquad \text{with} \quad \mathbf{p}(0) = \mathbf{p}_0.$$

(1) (5pts) Show

$$e^{\Omega t} = I_3 + \sin t \cdot \Omega + (1 - \cos t)\Omega^2.$$

(2) (5pts) Suppose that $\hat{\mathbf{w}} = (0, 1, 0)^T$ and $\mathbf{p}_0 = (0, 1, 1)^T$. What is the $\mathbf{p}(t)$ at $t = \pi$?