Algebra Qualifying Exam (2020 July)

1. (10 points) Prove that any finite group is isomorphic to a subgroup of a matrix group over any field.
2. (10 points) Determine whether a $\mathbb{R}$-linear map $T : \mathbb{R}^7 \to \mathbb{R}^7$ exists such that $T^2 + T + I$ is nilpotent.
3. (10 points) Let A be an $n \times n$-matrix over a field such that $A^k = I$ for some integer $k > 0$. Show that A is diagonalizable.
4. (10 points) Let G be a finitely generated group. For a finite subset $S \subset G$ which generates G, define

$$\begin{aligned} D_{G,S}(n) &= \{s_1 s_2 \cdots s_r \mid s_i \in S, s_i^{-1} \in S, r \leq n\} \\ \ell_{G,S}(k) &= \min\{n \mid \# D_{G,S}(n) \geq k\} \end{aligned}$$

   For any finite subsets $S, T \subset G$ generating G, show that

$$\limsup_{k\to\infty} \frac{\ell_{G,S}(k)}{\ell_{G,T}(k)} < \infty, \qquad \liminf_{k\to\infty} \frac{\ell_{G,S}(k)}{\ell_{G,T}(k)} > 0.$$

5. (10 points) Determine the degree of the field extension $\mathbb{Q}(\sqrt{65 + 56\sqrt{-1}})$ over $\mathbb{Q}$.
6. (10 points) Find the monic irreducible polynomial with rational coefficients which has as zero

$$\alpha = \frac{1 + \sqrt[3]{10} + \sqrt[3]{10^2}}{3}$$

7. (10 points) Let F be the finite field of order q. Then prove that

$$\#\mathrm{GL}_3(F) = (q^3 - 1)(q^3 - q)(q^3 - q^2).$$

8. (15 points) Let F be a finite field of order q.
   (a) Classify all the conjugacy classes in $\mathrm{GL}_3(F)$, whose characteristic polynomial is x^3 in terms of Jordan canonical forms.
   (b) Compute the size of each conjugacy class in the above question.
9. (15 points) Recall that a square matrix A is congruent to B if there exists an invertible square matrix P such that $PAP^{tr} = B$. We say that a square matrix A over $\mathbb{R}$ is hyperbolic if A is congruent (over $\mathbb{R}$) to a matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

   Determine the following matrices are hyperbolic with proof:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$