Qualifying Examination - Analysis - SU/2020.

1. Determine whether the following series converge or diverge.
   (1) (8pts) $\sum_{n=1}^{\infty} \frac{e^{in}}{n}$.
   (2) (7pts) $\sum_{n=1}^{\infty} \frac{1}{\sum_{k=1}^{n} \left[\frac{n}{k}\right]}$. (Here $[x]$ is an integer part of x.)

2. (10pts) Let $f(x)$ be a continuous function on $[0, \infty)$ such that $\lim_{x \to \infty} f(x) = L$ for some real number L. Show that

$$\lim_{n \to \infty} \int_0^1 f(nx)\, dx = L.$$

3. (10pts) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Suppose that

$$\int_0^1 f(x^{1/n})\, dx = 0 \quad \text{for all } n = 1, 2, \ldots.$$

Show that $f(x) = 0$ for all $x \in [0, 1]$.

4. (10pts) Prove or disprove:
   (1) (5pts) If $f : (0, \infty) \to \mathbb{R}$ is continuous and $f \in L^1(m)$, where m is the Lebesgue measure, then $\lim\limits_{x \to \infty} f(x)$ exists.
   (2) (5pts) There exists an enumeration of $r_n, n = 1, 2, \cdots$, of rationals such that

$$\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, r_n + \frac{1}{n} \right) \neq \mathbb{R}.$$

   (An enumeration is a complete, ordered listing of all elements in a collection.)

5. (10pts) Let f be an entire function for which there exists a positive number $M > 0$ and a polynomial P such that $|f(z)| \leq M|P(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = cP(z)$ for some constant c.

6. (10pts) Evaluate the integral

$$\int_0^{\pi} \frac{d\theta}{1 + a\cos\theta}$$

for $a \in (-1, 1)$.

7. (10pts) Find a conformal mapping which maps the domain D onto the open unit disk $\{z : |z| < 1\}$, where D is the intersection of $\{z : |z| < 1\}$ and the upper half plane $\{z : \text{Im } z > 0\}$.

8. Consider the following differential equation

$$y'' + \alpha y' + 4y = 0$$

where $y' = \frac{dy}{dt}$ and $y'' = \frac{d^2y}{dt^2}$.
(1) (7pts) For $\alpha = 5$ and $y(0) = 0, y'(0) = 3$, solve the initial value problem and describe how the solution behaves as $t \to \infty$.
(2) (8pts) For which values of α does the solution decay to 0 while oscillating?

9. (10pts) Find the solution of the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$