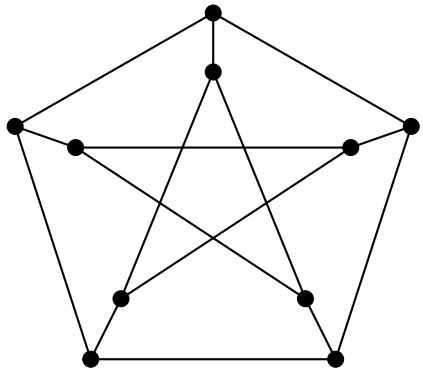
Discrete Math. Graduation Exam, Spring 2020

1. Show that the following graph is not planar.



2. Suppose there are n people numbered $1, 2, \ldots, n$. Also there are n hats numbered $1, 2, \cdots, n$. Let $\{D_n\}$ be the number of ways in which no one gets the hat having the same number as hers/his.

   (a) Show that $\{D_n\}$ satisfies the relation

   $$D_n = (n-1)(D_{n-1} + D_{n-2})$$

   for $n \geq 3$. *Hint.* Assume that the person 1 takes hat i. Now there are two possibilities, whether the person i takes hat 1 or not.

   (b) Show that $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$.

3. The XOR gate $\oplus$ is a function $\oplus : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ defined as

   $$x \oplus y := x\overline{y} \vee \overline{x}y.$$

   Express the following function $s : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}$ in terms of the XOR gate only.

x	y	z	$s(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0