Qualifying Examination - Analysis - 01/2020.

1. (10pts) Define a sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n = (-1)^k \quad \text{if } 2^k \le n < 2^{k+1}.$$

Is the following series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

2. (15pts) (1) Show that the function $f_1 : [0,1] \to [0,1]$ given by

$$f_1(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

is not Riemann-integrable.
(2) Show that the function $f_2 : [0,1] \to [0,1]$ given by

$$f_2(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{q}, & x = \frac{p}{q} (\neq 0) \text{ with } (p,q) = 1 \\ 0, & x : \text{irrational} \end{cases}$$

is Riemann-integrable. (Here, $(p,q) = 1$ means that p and q are relatively prime.)
(3) Prove or disprove:
If $f, g : [0,1] \to [0,1]$ are Riemann integrable, then $f \circ g$ is Riemann-integrable.

3. (10pts) If $f : [0,1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x^{1/n})\, dx = 0 \quad \text{for all } n = 1, 2, \ldots$$

then show that $f(x) = 0$ for all $x \in [0,1]$.

4. (10pts) A set $E \subset \mathbb{R}$ is said to be of *(Lebesgue) measure zero* if for any $\epsilon > 0$, there is a sequence of intervals $I_n = [a_n, b_n], n = 1, 2, \ldots$ such that

$$(1) E \subset \bigcup_n I_n \quad \text{and} \quad (2) \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

(1) Show that if A_n is of measure zero for all $n \in \mathbb{N}$, then $\bigcup_n A_n$ is of measure zero.
(2) Show that if Z is of measure zero, then $\{x^2 : x \in Z\}$ is of measure zero.

5. (10pts) Find all entire functions f such that $[f(z)]^2 = f(z^2)$ for all $z \in \mathbb{C}$.

6. (10pts) Show that if f is entire and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

7. (10pts) Evaluate the integral

$$\int_0^{\infty} \frac{x^{1/3}}{1+x^2} dx.$$

8. Consider the following initial value problem

$$y'' - 3y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = 2$$

(1) (8pts) Find the solution of the initial value problem above.
(2) (7pts) Determine the maximum value of the solution and also find the point where the solution is zero.

9. (10pts) Find out the general solutions of the following system

$$x' = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{pmatrix} x.$$