Algebra Qualifying Exam (2020 Jan 30)

Part I (Solve all the problems)

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

   Prove or disprove:

   (a) (5 point) For all 3×4 matrix B, there is a non-zero vector $x \in \mathbb{R}^4$ such that $ABx = 0$.

   (b) (5 point) For all 3×4 matrix B, there is a non-zero vector $y \in \mathbb{R}^3$ such that $BAy = 0$.

2. (10 points) Let $R = \mathbb{Q}[x]/(x^2 - 1)$. Find non-zero e and f in R such that

$$e^2 = e, \quad f^2 = f, \quad ef = 0, \quad e + f = 1.$$

   Show that the maps $p_e(r) = re$ and $p_f(r) = rf$ are ring homomorphisms of R to itself.

3. (10 points) Let $G = \mathrm{GL}_n(\mathbb{R})$ act on the set $S = \mathbb{R}^n$ by left multiplication.

   (a) (7 points) Describe the decomposition of S into orbits for this action. In other words, you must express S as a disjoint union of orbits.

   (b) (3 points) What is the stabilizer of the standard basis column vector $e_1 = (1, 0, \cdots, 0)$?

4. (15 points) Let G be the group given by the presentation $G = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$.

   (a) (7 points) Show that the abelianization of G is trivial.

   (b) (8 points) Show that G is non-trivial.

5. (15 points) Let n be a positive integer. Let F be a field of characteristic 0 which contains a primitive n-th root of unity. Show that any degree n cyclic extension of F is of the form $F(\sqrt[n]{a})$ for some $a \in F$.

6. (10 points) Let $\varphi : R \to R'$ be a nontrivial ring homomorphism and let P' be a prime ideal of R'. Assume that both R and R' contain $1 \neq 0$. Note that $\varphi(1) = 1$.

   (a) (7 points) Prove that $\varphi^{-1}(P')$ is a prime ideal of R.

   (b) (3 points) If $\varphi : \mathbb{Q} \to \mathbb{Q}[x]$ is defined by $\varphi(q) = q$, and $P' = (x - 1)$, then what is $\varphi^{-1}(P')$?

Part II (Solve your choices up to 30 points: Please choose 2 problems to be graded.)

7. (15 points) Let $\mathbb{F}_7$ be the finite field of order 7. Show that any two of the three rings

$$\mathbb{F}_7[x]/(x^2-5), \quad \mathbb{F}_7[x]/(x^2), \quad \mathbb{F}_7 \times \mathbb{F}_7$$

are not isomorphic.

8. (15 points) Let V be a finite-dimensional vector space over the real numbers $\mathbb{R}$.

(a) (7 points) If $\dim_{\mathbb{R}} V$ is odd, prove that every linear operator $A : V \to V$ has at least one real eigenvalue.

(b) (8 points) Suppose $A_1, A_2, \cdots, A_n$ are finitely many pairwise commuting linear operators on V. Assume that none of the operators A_i has a negative real eigenvalue. If the sum $A_1 + A_2 + \cdots + A_n$ is equal to the negative of the identity operator on V, show that $\dim_{\mathbb{R}} V$ is even. (Hint. Use induction on the dimension of V.)

9. (15 points) Let D_{2n} be the dihedral group of order $2n$ and let G be a finite group. Assume that there is a non-trivial homomorphism $f : D_{2n} \to G$. Then show that the order of G is even.

10. (15 points) Consider the function fields $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ of one variable, and regard L as a finite extension of K via the $\mathbb{C}$-algebra inclusion

$$x \to -\frac{(y^5-1)^2}{4y^5}$$

Show that the extension L/K is Galois and determine its Galois group.