Qualifying Examination - Analysis - 07/2019.

1. Determine whether the following series are convergent or divergent.

(1) (8pts) $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

(2) (7pts) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$

2. (10pts) Let $f(x)$ be a differentiable function on $[0,1]$. Suppose that $f'(0) < \alpha < f'(1)$. Show that there is $x_0 \in (0,1)$ such that $f'(x_0) = \alpha$.

3. (10pts) Suppose that $f(x)$ is a continuous function and $f(x+1) = f(x)$ for all x. Show that the following sequence

$$g_n(x) := \frac{1}{n} \sum_{i=1}^{n} f\left(x + \frac{i}{n}\right)$$

converges uniformly on $[0,1]$.

4. (10pts) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers. Show that the following are equivalent.

- $\liminf_{n\to\infty} a_n = 0$
- There is an enumeration $(r_n)_{n\in\mathbb{N}}$ of rational numbers (that is, $\{r_n : n \in \mathbb{N}\} = \mathbb{Q}$) such that

$$\bigcup_{n\in\mathbb{N}} (r_n - a_n, r_n + a_n) \neq \mathbb{R}.$$

5.

(a) (7pts) Solve the following differential equation:

$$y' = \frac{-x+y}{2x+y}.$$

(b) (8pts) Solve the following initial value problem:

$$y''' + y' = t^2, \quad y(0) = y''(0) = 0, \quad y'(0) = 1.$$

6. (10pts) Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

7. (10pts) Prove or disprove each of the following statements.

(a) There exists an entire function f such that $f(1/n) = (1/n^2)i$ and $f(i) = 1$ for all $n \in \mathbb{N}$.

(b) There exists an entire function f such that $f(n) = n^2 i$ and $f(i) = i$ for all $n \in \mathbb{N}$.

8. (10pts) Let Ω be a connected domain. Suppose that $f : \Omega \to \mathbb{R}$ is a continuous function satisfying

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta,$$

whenever $\overline{D_r} = \{z : |z - a| \leq r\} \subset \Omega$. If f attains a maximum in Ω, then f is constant.

9. (10pts) Evaluate the integral

$$\int_0^\infty \frac{1 - \cos 2x}{x^2} dx.$$