Algebra Qualifying Exam (4 hours)

July 25, 2019

PART I (Solve all problems)

1. (10 points) Let P be a prime ideal of the integral ideal R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial in $R[X]$ (here $n \geq 1$). Suppose $a_{n-1}, \cdots, a_1, a_0$ are all elements of P and suppose a_0 is not an element of P^2. Show that $f(x)$ is irreducible in $R[X]$.

2. (10 points) Let m and n be integers. Give a formula for an isomorphism of abelian groups

$$\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/gcd(m,n)\mathbb{Z} \oplus \mathbb{Z}/lcm(m,n)\mathbb{Z}.$$

3. (20 points) Answer the following questions.

(a) Show that every group of order $5 \cdot 13$ is cyclic.

(b) Prove or disprove: every group of order $3 \cdot 5 \cdot 17$ is cyclic.

4. (15 points) Answer the following questions.

(a) Determine the cardinality of the algebraic closure of $\mathbb{Q}$.

(b) Prove or disprove: There exists a field whose algebraic closure is finite.

5. (15 points) Answer the following questions.

(a) Show that $f(x^c)$ is irreducible over $\mathbb{Q}$ for all integers $c \geq 1$, where $f(x) = -3x^2 + 7x - 3$.

(b) Find an irreducible polynomial $p(x) \in \mathbb{Z}[X]$ such that $p(1) = 1, p(x^{-1}) \cdot x^{\text{deg p}} = p(x)$, and $p(x^2)$ is not irreducible.

PART II (Solve your choices up to 30 points)
채점방식: 각 소문제를 채점하여 상위 두 문제 점수를 합산함

6. (15 points) Let k be a field of characteristic 0. Let f be an irreducible polynomial in $k[x]$. Prove that f has no repeated factors, even over an algebraic closure of k.

7. (15 points) Let $T \in \text{Hom}_{\,k}(V)$ for a finite-dimensional k-vector space V, with k a field. Suppose that T is diagonalizable on V. Show that $k[T]$ contains the projectors to the eigenspaces of T .

8. (15 points) Answer the following questions.

(a) There are two ways to define the lower central subgroup G_k of a group G:

$$G_1 = G, \text{ and either } G_k = [G, G_{k-1}] \text{ or } G_k = [G_{k-1}, G] \text{ for } k \geq 2.$$

Prove or disprove: Are these equivalent?

(b) Let Γ be an infinite cyclic group with a generator t. Let $A = \mathbb{Z}$ on which Γ acts by $t \cdot n = -n (n \in A)$. Let $G = A \rtimes \Gamma$ be the semi-direct product. Compute G_k for every $k \geq 1$ and compute $\bigcap_{k \geq 1} G_k$.

9. (15 points) Answer the following questions.

(a) Prove or disprove: Every subgroup of a finitely generated group is finitely generated.

(b) Suppose $F : G \to \Gamma$ is a group epimorphism with G finitely generated and Γ finitely presented. Show that $H = \ker f$ is the normal closure of a finite subset in H.

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