MATH 351 Introduction to Numerical Analysis

Graduation Exam
May, 2019

1. (10 points) The equation $x^2 - a = 0$ with $a > 0$ has a unique positive root $\sqrt{a}$. There are many ways to change the equation to the fixed-point form

$$x = g(x).$$

For the following four functions

$$\begin{aligned} x &= g_1(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \\ x &= g_2(x) = \frac{x(x^2+3a)}{3x^2+a}, \\ x &= g_3(x) = \frac{a}{x}, \\ x &= g_4(x) = 2x - \frac{a}{x} \end{aligned}$$

discuss the convergence (or nonconvergence) behaviour of the iteration $x_{n+1} = g(x_n), n = 0, 1, 2, \ldots$ with $x_0 > 0$. Determine the order of the convergence if it converges.

2. (10 points) Let $p_3(x)$ be the unique polynomial of degree at most 3 that interpolates $f(x) = \left[\cos\left(\frac{\pi x}{2}\right)\right]^2$ at four points $-1, 0, 1$ and 2.

a) Form the divided difference table for f at these points and write $p_3(x)$ in its Newton form.

b) Write $p_3(x)$ in its Lagrange form.

c) Estimate $f\left(\frac{1}{2}\right)$.

d) Estimate the maximum possible error of the answer in c).

3. (10 points) Use the method of undetermined coefficients to find a second-order accurate finite difference approximation to $f''(x)$ based on 3 equally spaced points $x - h, x, x + h$

$$f''(x) = a_0 f(x-h) + a_1 f(x) + a_2 f(x+h) + O(h^2).$$