Qualifying Examination - Analysis - 01/2019

1. (a) (8pts) Let

$$s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

Show that $\lim_{n\to\infty}(s_n - 2\sqrt{n})$ exists.

(b) (7pts) Let

$$S = \{a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + \cdots + a_k \cdot 3^k | a_i = 0 \text{ or } 2, k = 0, 1, 2, \dots\}.$$

Define

$$b_n = \begin{cases} 0, \, n \notin S \\ 1, \, n \in S. \end{cases}$$

Show that $\sum \frac{b_n}{n}$ converges.

2. (10pts) Let $f(x)$ be a continuous function on $[0, 1]$. Show that

$$\lim_{p\to\infty} \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} = \sup_{x\in[0,1]} |f(x)|.$$

3. (10pts) Suppose that f_n and g_n are defined on $[0, 1]$, and

(1) $\sum f_n$ has uniformly bounded partial sums;

(2) $g_n \to 0$ uniformly on $[0, 1]$;

(3) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in [0, 1]$.

Show that $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on E.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f_n : X \to [0, \infty]$ is a sequence of measurable functions satisfying $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

(a) (5pts) Prove that if $f_1 \in L^1(\mu)$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

(b) (5pts) Show that the above conclusion does not hold if the condition $f_1 \in L^1(\mu)$ is omitted.

5. Solve the following initial value problems:

(a) (7pts) $(x+1)y' + y^2 = y, \quad y(0) = \frac{1}{2}$.

(b) (8pts) $xyy' = x^3 + y^2, \quad y(1) = 2$.

6. (10pts) Solve the following initial value problem:

$$y'' - y' - 2y = 3e^{2t}, \quad y(0) = 1, \quad y'(0) = 2.$$

7. (10pts) Suppose that f and $g_1,\ g_2,\ g_3, \ldots$ are entire functions. Assume that $|g_n^{(k)}(0)| \leq |f^{(k)}(0)|$ for all k and n, and also assume that $\lim_{n\to\infty} g_n^{(k)}(0)$ exists for all k. Show that the sequence $\{g_n\}$ converges uniformly to an entire function on each compact set.

8. (10pts) Evaluate $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\, dx$.

9. (10pts) Find a conformal mapping which maps the domain D onto the open unit disc $\{z : |z| < 1\}$, where D is the intersection of $\{z : |z| < 1\}$ and the upper half plane $\{z : \text{Im } z > 0\}$.