Qualifying Exam: Real Analysis - 06/2018

1. (10pts) Let $\mathcal{M}$ be an infinite σ-algebra. Show that $\mathcal{M}$ is uncountable.

2. (10pts) Let (A_n) be a sequence of measurable sets in a measure space $(X, \mathcal{M}, \mu)$ such that $A_n \subset A_{n+1}$ and let $A = \cup A_n$. Show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

3. (15pts) For $f \in L^1(\mathbb{R}, m)$, (m is Lebesque measure), define $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Show that F is uniformly continuous.

4. (15pts) Let $f : \mathbb{R}^n \to \mathbb{C}$ be in L^1_{loc}. Define the Hardy-Littlewood maximal function Hf by

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|\,dy,$$

where m is the Lebesque measure on $\mathbb{R}^n$.
Show that there is a constant $C > 0$ such that for all $f \in L^1$ and all $\alpha > 0$,

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f(x)|\,dx.$$

5. (10pts) Let m be the Lebesque measure on $[0, 1)$. Show that for m-a.e. x, the decimal expansion of x contains all the digits $0, 1, \ldots, 9$.

6. (10pts) Let (a_n) and (b_n) be sequences of positive real numbers such that $a_n b_n \geq 1$ for all n. Show that

$$\sum_{n=1}^{\infty} (\frac{a_n}{2^n}) \, (\sum_{n=1}^{\infty} \frac{b_n}{2^n}) \geq 1.$$

7. (15pts) Let $X = [0, 1]$, with the Lebesque measure μ. For g in L^2, define a bounded linear functional ϕ_g by

$$\phi_g(f) = \int fg\,d\mu.$$

(a) Show that $\|\phi_g\| = \|g\|_2$.
(b) Show that $\lim_{n\to\infty} \phi_g(\sin 2\pi nx) = 0$.

8. (10pts) Show that L^1 is complete.