Probability Theory Qualifying Examination
June 2018 POSTECH

(5 problems. Justify all your work.)

1. (20 pts) Assume that all random variables in this problem are defined on the same probability space.

   (a) Suppose that X_n converges in distribution to X and X_n converges to Y almost surely. Prove or disprove by a counter example that $X = Y$ a.s.

   (b) Suppose that X_n converges in probability to X and X_n converges to Y almost surely. Prove or disprove by a counter example that $X = Y$ a.s.

2. (20 pts) Suppose that $X_1, X_2, \ldots$ are uncorrelated random variables such that $0 \le X_i \le 1$ with probability 1 for all $i \ge 1$ and let $S_n = X_1 + \cdots + X_n$. Assume that $\mathbf{E}S_n \to \infty$ as $n \to \infty$. Show that

$$\frac{S_n}{\mathbf{E}S_n} \to 1 \quad \text{almost surely.}$$

3. (20 pts) Suppose X_n converges in distribution to X and $Y_n \ge 0$ converges in distribution to some constant $c > 0$. Show that $X_n Y_n$ converges in distribution to cX.

4. (20 pts) Let $X_1, X_2, \ldots$ be random variables such that $S_n := X_1 + \cdots + X_n$ is a martingale with respect to the filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. Show that

$$E[X_i X_j] = 0 \quad \text{if } i \ne j.$$

5. (20 pts) Let ξ_i^n be independent and identically distributed nonnegative integer-valued random variables. Define a sequence Z_n by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0. \end{cases}$$

   Assume that $\mathbf{P}(\xi_i^n = 0) > 0$ and $\mathbf{E}(\xi_i^n) > 1$. Define, for $s \in [0, 1]$,

$$\varphi(s) := \mathbf{E}s^{\xi_i^n} = \sum_{k=0}^{\infty} s^k p_k,$$

   where $p_k := \mathbf{P}(\xi_i^n = k)$. It is known that there exists a unique $\rho \in (0, 1)$ such that $\varphi(\rho) = \rho$.

   (a) Show that $M_n := \rho^{Z_n}$ is a positive and bounded martingale with respect to the filtration $\mathcal{F}_n := \sigma(\xi_i^m, i \ge 1, 1 \le m \le n)$.

   (b) Let $N := \inf\{n \ge 0 : Z_n = 0\}$. Show that $\mathbf{P}\{N = \infty\} > 0$.