Qualifying Examination - Analysis - 07/2018

1. (a) (8pts) Evaluate

$$\lim_{n\to\infty} \frac{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}}{\sqrt{n}}.$$

(b) (7pts) Determine whether the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{e^{in}}{\log n}$$

2. (10pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there is $r \in (0,1)$ such that $|f'(x)| \le r$ for all x. Let $a_1 \in \mathbb{R}$ and define $a_{n+1} = f(a_n)$. Show that the sequence (a_n) converges.

3. (10pts) Let (x_n) be a sequence in $[0,1]$ such that for any $h \in \mathbb{Z}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0 \end{cases}.$$

Show that for any continuous function f on $[0,1]$ with $f(0) = f(1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx.$$

4. (a) (5pts) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $\{E_n\}_{n=1}^{\infty}$ is a countably family of measurable sets and that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Show that the set $E = \{x \in X : x \in E_n \text{ for infinitely many } n\}$ is measurable and $m(E) = 0$.
(b) (5pts) Let d be the Euclidean metric on $\mathbb{R}^3$ and let m be the Lebesque measure on $\mathbb{R}^3$. Let (A_n) be a sequence of sets such that $A_n \subset [0,1]^3$ and $|A_n| = n$ ($|A_n|$ = number of elements in A). Show that m-a.e. point $x \in [0,1]^3$ satisfies that $d(x, A_n) > \frac{1}{n}$ for all but finitely many times.

5. (10pts) Show that if f is entire and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

6. (10pts) Evaluate $\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx$.

7. (10pts) Find a conformal mapping which maps the domain D onto the open unit disc, where D is the intersection of $|z| < 1$ and $|z - 1| < 1$.

8. (10pts) Find out the general solutions of the following system

$$x' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} x\,.$$

9. Consider the following initial-value problem:

$$2y'' - 3y' + y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{2}$$

(a): (8pts) Find the solution of the initial value problem above.

(b): (7pts) Determine the maximum value of the solution and also find the point where the solution is zero.