Algebra Qualifying Exam (4 hours)

July 26, 2018

PART I (Solve all problems)

1. (15 points) Let F be a field. Show that every finite subgroup of the multiplicative group F^* is cyclic.

2. (15 points) Determine the order of the group $\mathrm{PSL}_2(\mathbb{F}_7)$. Here $\mathrm{PSL}_2(\mathbb{F}_7)$ is the quotient group of the 2×2 special linear group defined over $\mathbb{F}_7$ by its center.

3. (15 points) Let $m_1, m_2, \cdots, m_n$ be positive integers which are pairwise relatively prime. Show that

$$\mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \cdots, \sqrt{m_n}).$$

4. (15 points) Let $A \in M_n(\mathbb{Q})$ with $A^k = I_n$. If j is a positive integer with $(j,k) = 1$, show that $tr(A) = tr(A^j)$. Does the same statement hold if $A \in M_n(\mathbb{C}) \setminus M_n(\mathbb{Q})$?

5. (5 points each) Compute the Galois group of the following polynomial over $\mathbb{Q}$:

$$(a)\ x^3 - 2x + 2,\ \ (b)\ x^3 - x^2 - 1,\ \ (c)\ x^4 - 4x^2 + 5,\ \ (d)\ x^4 - 5x^2 + 5.$$

PART II (Solve your choices up to 20 points)

6. (10 points) Let G be a group of order $p^n q^m$ for primes p, q with $p < q$ and assume that the order of $[p] \in (\mathbb{Z}/q\mathbb{Z})^\times$ is larger than n. Show that there are subgroups

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{n+m} = G$$

where each H_j is normal in H_{j+1} and H_{j+1}/H_j is cyclic of prime order.

7. (10 points) Let $M_n(\mathbb{Z})$ be the ring of $n \times n$ matrices whose entries are integers. Find all maximal (two-sided) ideals of $M_n(\mathbb{Z})$. For each maximal ideal $\mathfrak{m}$, what is the number of elements in $M_n(\mathbb{Z})/\mathfrak{m}$?

8. (10 points each) Let M be a finitely generated $F[x]$-module where F is a field.

(a) Show that if $f(x)m = 0$ for $f(x) \neq 0$ forces $m = 0$, then M is a projective $F[x]$-module.

(b) Let H be an $F[x]$ submodule of M. Show that if $f(x)m \in H$ for $f(x) \neq 0$ implies $m \in H$, then $M \simeq H \oplus K$ for a submodule K of M.

The End