Discrete Mathematics Graduation Exam

(You must include justifications in order to receive full credits.)

1. Let $\mathcal{U}$ be the collection of all infinite sets. Consider the equivalence relation $\sim$ defined by

$$A \sim B \leftrightarrow \exists f : A \to B \text{ one to one and onto}$$

for $A, B \in \mathcal{U}$. We write $[A]$ for the equivalence class containing $A \in \mathcal{U}$.

   (a) (3pts) Show that the following operation is ill-defined by giving a specific example.

$$[A] \oplus [B] := [A \cup B]$$

   (b) (3 pts) Show that the following operation is well-defined.

$$[A] \boxplus [B] := [A \times \{1\} \cup B \times \{0\}]$$

   (c) (4 pts) Give an example of a set A such that $[A] \boxplus [A] = [A]$.

2. (a) (4 pts) Give an example of a Boolean algebra.

   (b) (5 pts) Determine whether the following combinatorial circuits are equivalent.

$$\overline{(\bar{x_1} \wedge x_2) \vee (x_1 \wedge \bar{x_3})}, \quad (x_1 \vee \bar{x_2}) \wedge (x_1 \vee \bar{x_3})$$

3. (7 pts) Using the definition of a tree as a connected graph without cycles, show that the two statements below are equivalent. Let T be a graph with n vertices. a) T is a tree, b) T is a graph with $n-1$ edges and without cycles.

4. Consider the following algorithm. Input: A connected, weighted graph (all weights are positive) and a particular vetex a.

$L(a) := 0 \quad , L(x) := \infty, \text{ for } x \neq a$

$T :=$ all vertices

while $T \neq \emptyset$

choose $v \in T$ with minimum L value

for each $x \in T$ adjacent to v

$L(x) := \min\{L(x), L(v) + w(v,x)\}$

$T := T \setminus \{v\}$

   (a) (7pts) Show that when the algorithm terminates, for each vertex x, $L(x)$ is the weight of a shortest path from a to x.

   (b) (4 pts) Show that the worst case run time for a graph with n vertices is $\Theta(n^2)$.