Ph.D. Qualifying Examination. Real Analysis. June 2017, Postech

4 Problems. Justify all your work.

1. (25 = 10 + 15 points) (a) Show that the outer measure $m^*(E)$ $(= |E|_e)$ is (countably) subadditive on $\mathbb{R}^n$.

(b) Show that Lebesgue measure dm (with $m(E) = |E| = \int_E dm = \int_E dx$ for a measurable set $E \subset \mathbb{R}^n$) is countably additive on the σ-algebra Σ of (Lebesgue) measurable sets E in $\mathbb{R}^n$. (First show that if any two sets A and $B \subset \mathbb{R}^n$ satisfy $d(A, B) > 0$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.)

2. (25 = 13+12 points) (a) Let f, g, f_n, $g_n \in L^1(\mathbb{R}^d)$, $|f_n| \leq g_n$, and suppose that $f_n \to f$ a.e. and $g_n \to g$ a.e. If $\int g_n \to \int g$, show that $\int f_n \to \int f$.

(b) Let $1 \leq p < \infty$. Assume that f_n, $f \in L^p(\mathbb{R}^d)$ and $f_n \to f$ a.e. If $\|f_n\|_p \to \|f\|_p$, then show that $\|f_n - f\|_p \to 0$, as $n \to \infty$.

3. (25 = 15+10 points) (a) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Prove Jensen's inequality:

$$\phi\left(\int_{\mathbb{R}^n} f(x)\, g(x)\, dx\right) \leq \int_{\mathbb{R}^n} \phi(f(x))\, g(x)\, dx$$

if f is any real-valued integrable function on $\mathbb{R}^n$ and $g \geq 0$ is a real-valued integrable function on $\mathbb{R}^n$ such that $\int g = 1$.

(b) Use (a) to show that

$$\exp(\lambda_1 a_1 + \cdots + \lambda_n a_n) \leq \lambda_1 e^{a_1} + \cdots + \lambda_n e^{a_n}$$

whenever $a_j \in \mathbb{R}$, $\lambda_j \in [0, 1]$, $\lambda_1 + \cdots + \lambda_n = 1$, $n \geq 1$.

4. (25 = 10 + 15 points) Suppose that f is an absolutely continuous function on $[0, 1]$. (a) Show that f is of bounded variation. (b) Show also that f' exists a.e., f' is integrable on $[0, 1]$ and satisfies

$$\int_0^x f'(y)\, dy = f(x) - f(0)$$

for $0 \leq x \leq 1$.