**Qualifying Exam** **Numerical Analysis** **June, 2017**

1. (20 points) We need to compute a numerical value of $\pi$ by using either $f(x) = 4\tan^{-1}(x)$ or $g(x) = 4\sin^{-1}(x)$ since

$$4\tan^{-1}(1) = \pi, \qquad 4\sin^{-1}(1/\sqrt{2}) = \pi.$$

   (a) Compute the condition numbers $cf$ and $cg$ of $f(1)$ and $g(1/\sqrt{2})$, respectively.

   (b) Choose either $f(1)$ or $g(1/\sqrt{2})$ as a numerical value of $\pi$ and explain the reason.

2. (20 points) Let $f(x) = \sin(\pi x)$. Consider the uniform nodal points $x_k = (1/n)k$ for $k = 0, 1, \ldots, n$ of the interval $I = [0, 1]$. Let $P_n(x)$ be the piecewise cubic polynomial such that

   (a) $P_n(x)$ is a cubic polynomial on $[x_k, x_{k+1}]$.

   (b) $P_n(x_k) = f(x_k)$ and $P_n'(x_k) = f'(x_k)$ for $k = 1, 2, \ldots, n$.

   Find an upper bound $M(n)$ of the error

$$|P_n(x) - f(x)| \leq M(n) \qquad \text{for each } x \in I$$

   and prove that

$$\lim_{n\to\infty} P_n(x) = f(x) \qquad \text{for each } x \in I.$$

3. (20 points) Let $p_k(x)$ be the monic Chebyshev polynomial of degree $k = 0, 1, \ldots,$ and let $x_1, x_2, \ldots, x_n$ be the zeros of $p_n$.

   (a) Prove that the matrix

$$A = \begin{bmatrix} p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ \vdots & \cdots & \cdots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{bmatrix}$$

   is nonsingular.

   (b) Let $Q(f)$ be the Gaussian quadrature rule with respect to the zeros $x_1, x_2, \ldots, x_n$:

$$Q(f) = \sum_{j=1}^{n} f(x_j)w_j \approx \int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}dx.$$

   Show that all weights $w_1, w_2, \ldots w_n$ are equal to a number $c$, and find the number $c$.

4. (20 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $A^+$ is the pseudo-inverse of $A$.

5. (20 points) Prove that a positive definite matrix is invertible and its eigenvalues are all positive.

The End