Complex Analysis Qualifying Exam

Summer 2017

1. (10 points) Show that the harmonic function $u(z) := u(x,y) = \mathrm{Log}\sqrt{x^2+y^2} = \mathrm{Log}\,|z|$ has no harmonic conjugate in $\mathbb{C}\setminus\{0\}$.

2. (10 points) By integrating $(1-e^{2iz})/_{z^2}$ along an appropriate contour, show that

$$\int_0^\infty \frac{\sin^2 x}{x^2} = \frac{\pi}{2}.$$

3. (10 points) Show that if $0 < a < 1$, then

$$\lim_{R\to\infty}\int_{1-iR}^{1+iR}\frac{a^{-z}}{z}\,dz = 2\pi i$$

4. (10 points) Show that if $f: D(0,1) \to \mathbb{C}$ is holomorphic, then

$$|f(0)| \le \frac{1}{\sqrt{\pi}}\left(\int_{D(0,1)}|f(x,y)|^2 dxdy\right)^{1/2}$$

5. (10 points) Suppose $f(z)$ and $g(z)$ are holomorphic inside a simple closed contour C. Also suppose that f, g are continuous in the region consisting of C and its interior. Show that if $\mathrm{Re}\, f(z) = \mathrm{Re}\, g(z)$ on C, then $f(z) = g(z) + i\alpha$, inside C, where α is a real constant.

6. (10 points) Let $\lambda > 2$ be a number. Show that the equation

$$ze^{\lambda - z} = 2$$

has a unique solution in $\{z: |z| < 2\}$.

7. (10 points) Suppose that u is bounded and harmonic in $\{(x,y): 0 < x^2 + y^2 < 4\}$. Show that u has a unique harmonic extension to $D(0,2)$.

8. (10 points) Suppose $f: D(0,R) \to \mathbb{C}$ is holomorphic and $|f(z)| \le M < \infty$. Show that for $|z| = r < R$,

$$|f(z) - f(0)| \le 2\frac{Mr}{R}.$$

9. (10 points) Show that $\sum b_j$ converges absolutely if and only if $\prod(1 + b_j)$ converges absolutely.

10. (10 points) Let $\{f_n(z)\}$ be a sequence of holomorphic functions defined on an open set U. Suppose that

$$\prod_{j=1}^{n} f_j(z) \to f(z)$$

uniformly on compact subsets in U. Show that

$$\sum_{k=1}^{n} f_k'(z) \prod_{j \ne k} f_j(z) \to f'(z)$$

uniformly on compact subsets in U.