Algebra II Qualifying Exam
2017 Spring

1. (20pts) Suppose $f(x)$ is an irreducible polynomial over $\mathbb{Q}$ with degree p prime, and $f(x)$ has exactly two non-real roots. Let F be a splitting field of $f(x)$ over $\mathbb{Q}$.
   (a) Show that $[F : \mathbb{Q}]$ is a multiple of p.
   (b) Show that the Galois group of $F/\mathbb{Q}$ is the symmetric group S_p on p letters.

2. (20pts) Let F be the splitting field of $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1$ over $\mathbb{Q}$. Let ζ be a zero of $f(x)$.
   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
   (b) Show that the Galois group of F over $\mathbb{Q}$ is cyclic.
   (c) Let $\alpha = \zeta + \zeta^2 + \zeta^4$. Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.
   (d) Let $\beta_k = \zeta + \zeta^k$. Find an integer k such that $[\mathbb{Q}(\beta_k) : \mathbb{Q}] = 3$. In this case, is $\mathbb{Q}(\beta_k)$ Galois over $\mathbb{Q}$?

3. (20pts) Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$.
   (a) Show that there is a maximal subfield F of $\overline{\mathbb{Q}}$ not containing $\sqrt{2}$.
   (b) Show that a finite extension E of F is cyclic. (Hint: first consider the case that E is Galois.)

4. (20pts) Suppose V is a free module over a commutative ring R with unity, and $f\colon V \to V$ is an endomorphism. Let A be the matrix associated to f with respect to a fixed basis for V. Show that $(\det A) \cdot x = 0$ for each element x in the cokernel of f.

5. (20pts) Show that for every square matrix A over an algebraically closed field, the transpose A^T is similar to A, that is, $A^T = PAP^{-1}$ for some nonsingular P. Is it still true if the field is not assumed to be algebraically closed?