1. (20 points) In this problem, you should use only 4 decimal-digit arithmetics with rounding-up. Let $a = 15.6$, $b = 15.7$, $\theta = \frac{1}{\pi}$. In the following evaluation, you may use the valus

$$\pi = 3.142, \quad \theta = 0.3183, \quad cos\theta = 0.9498, \quad sin\theta = 0.3313.$$

   (a) Evaluate $f(a, b, \theta) = a^2 - 2ab\cos\theta + b^2$ and denote it by y_1

   (b) Change the expression of $f(a, b, \theta)$ into a sum of two positive terms. Evaluate the expression and denote it by y_2.

   (c) The more exact value y_e of $f(a, b, \theta)$ obtained by 8-decimal-digit arithmetics is $y_e = 24.62$. Between y_1 and y_2, which is closer to y_e. Discuss the reason why this phenomena happens.

2. (20 points) Let $f(0) = 0, f(1) = \frac{1}{4}, f(2) = 1, f(3) = \frac{1}{4}, f(4) = 1$. Find the natural cubic spline $S(x)$ which interpolates f at $x = 0, 1, 2, 3, 4$, with $S''(0) = S''(4) = 0$.

3. (20 points) Let $\{p_0, p_1, \ldots, p_n\}$ be a set of orthogonal polynomials with leading coefficient with 1, with respect to the inner product

$$(f, g)_w = \int_a^b f(x)g(x)w(x)dx,$$

where $w(x) \geq 0$ is an weight function, *i.e.*, the interval (a, b) does not contain any interval (α, β) where $w(x)$ is 0 identically.

Prove that p_n has all simple roots $x_1, x_2, \ldots, x_n$ in the open interval (a, b).

4. (20 points)

   (a) Write down the statement of Gershgorin's circle theorem to find some region which contains all eigenvalues of the $n \times n$ matrix $A = [a_{ij}]$.

   (b) Prove Gershgorin's circle theorem.

5. (20 points) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

   (a) Find the lower triangula matrix L such that $A = LL^T$.

   (b) Compute the number $cond(A)$ which is the the condition number of A with respect to the l_∞- vector norm.

The End