1. [20] Consider a decision problem with $\theta \in \{-1, 0, 1\}$. A random variable X is observed from a normal distribution with density $\phi(x-\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$, $x \in \mathbb{R}$. We consider decision rules $\delta(x) = -1, 0, 1$ according as $x \in B_-, B_0, B_+$, respectively, where B_-, B_0, B_+ are disjoint subsets of $\mathbb{R}$ such that $B_- \cup B_0 \cup B_+ = \mathbb{R}$. The loss function is given by $L(\theta, \delta) = |\theta - \delta|$. We further suppose that the prior distribution of θ is given by $\varphi(-1) = \varphi(1) = q$ and $\varphi(0) = 1-2q$ for some $0 < q < 1/2$.

(a) [5] Obtain the posterior distribution $\varphi(\theta|x)$.

(b) [8] Calculate the posterior risk $r(x) = E[L(\theta, \delta(x))|x]$, and verify that the Bayes rule is given by

$$\delta^*(x) = \begin{cases} 1, & \text{if } x > k, \\ 0, & \text{if } -k \leq x \leq k, \\ -1, & \text{if } x < -k, \end{cases}$$

where $k = \sinh^{-1}\left(\frac{1-2q}{2q}\sqrt{e}\right)$.

(c) [7] Write down the risk function $R(\theta, \delta^*)$ for $\theta \in \Theta$, and find a minimax rule. (Hint: $\Phi(-1) = 0.159$.)

2. [15] Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \ \lambda > 0.$$

(a) [5] What is the Cramer-Rao lower bound for unbiased estimators of $\theta = e^{-\lambda} = P(X_1 = 0)$?

(b) [10] Obtain a uniformly minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ for θ.

3. [15] (Continued from the previous problem.)

(a) [5] Find a maximum likelihood estimator (MLE) $\hat{\theta}$ for θ, and show that it is biased.

(b) [5] What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

(c) [5] Compare the asymptotic efficiency of the UMVUE $\tilde{\theta}$ to that of the MLE $\hat{\theta}$. Which one do you prefer?

– Continued on the next page –

4. [20] Let $X_1, \ldots, X_n$ be a random sample from a population with distribution function

$$F(x;\theta,\tau) = \begin{cases} 1, & x \geq \tau, \\ (x/\tau)^\theta, & 0 < x < \tau, \\ 0, & x \leq 0, \end{cases}$$

for $\theta > 0$, and $\tau > 0$.

(a) [10] Find thr MLE for θ and τ.

(b) [10] Are the MLEs consistent?

5. [15] The size of crucian carpes in a pond is thought to follow an exponential distribution with density

$$f(x;\mu) = \frac{1}{\mu} e^{-x/\mu}, \quad x > 0,\ \mu > 0.$$

To estimate mean μ (in centimeters), fishes were netted and measured as $X_1, X_2, \ldots, X_n$, assumed to be independent. *However, the net can catch only the fishes that are larger than 8 centimeters.*

(a) [5] Find the MLE for μ and show that it is unbiased.

(b) [10] Construct a uniformly most powerful test of level α, $0 < \alpha < 1$, for $H_0 : \mu \leq 6$ vs $H_1 : \mu > 6$, using a chi-square distribution for reference.
(Hint: $\mathcal{X}_2^2$ distribution is the same as exponential distribution with mean 2.)

6. [15] Let (X_i, Y_i), $i = 1, \ldots, n$, be a random sample of pairs from a bivariate normal distribution with mean $(\mu, -\mu)$, unit variances and correlation of $1/2$. It has density

$$f(x,y;\mu) = \frac{1}{\sqrt{3}\pi} \exp\left[-\frac{2}{3}\left\{(x-\mu)^2 - (x-\mu)(y+\mu) + (y+\mu)^2\right\}\right],$$

for real x, y, μ. Find a uniformly most powerful level α $(0 < \alpha < 1)$ test for $H_0 : \mu \leq 0$ vs $H_1 : \mu > 0$.

– The end –