Qualifying Examination - Complex Analysis - 01/2018

1. Prove or disprove the following statements.

(a, 6pt) There is a holomorphic function f in the open unit disc $D = \{|z| < 1\}$ such that $f(1/n) = 1/n^2$ for every positive integer n, and $f(i/2) = 1/4$.

(b, 6pt) There is a nonzero holomorphic function f in the open unit disc $D = \{|z| < 1\}$ such that $f(1 - \frac{1}{n}) = 0$ for every positive integer n.

(c, 6pt) There exists an entire function f such that $\lim_{|z|\to\infty} |f(z)| = 1$ and $f(0) = 0$.

(d, 6pt) There exists a sequence of polynomials P_n such that $P_n(0) = 1$ for $n = 1, 2, 3, \ldots$, but $P_n(z) \to 0$ for every $z \neq 0$, as $n \to \infty$.

(e, 6pt) If f is holomorphic in an open set Ω containing the closed unit disc $\overline{D}$, $|f(z)| > 1$ for $|z| = 1$, and $f(0) = 1$, then f has a zero in the open unit disc D.

2 (10pt). Let $\phi(z)$ be a continuous function along an arbitrary contour L of finite length and

$$f(z) = \frac{2!}{2\pi i} \int_L \frac{\phi(\zeta)}{(\zeta - z)^3} d\zeta \ (z \in \mathbb{C} \setminus L).$$

Give a direct proof of the statement : f is differentiable on $\mathbb{C} \setminus L$ and

$$f'(z) = \frac{3!}{2\pi i} \int_L \frac{\phi(\zeta)}{(\zeta - z)^4} d\zeta.$$

3 (10pt). Show that if $f : \mathbb{C} \to \mathbb{C}$ is a function such that both f^2 and f^3 are entire functions, then f is also entire.

4. (a, 10pt) Suppose that f is holomorphic on the open unit disc D and that $f(z) \neq 0$ for all $z \in D$. Show that there is a holomorphic function g on D such that $f(z) = e^{g(z)}$.

(b, 5pt) Is the conclusion of (a) still valid, if D is replaced by an arbitrary connected open set in the complex plane ?

5 (10pt). Let D be the open unit disc. Show that if $f = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on D and $|f(z)| \leq M$ for all $z \in D$, then $M|a_1| \leq M^2 - |a_0|^2$.

6 (15pt). Evaluate the integral

$$\int_0^\infty \frac{\log x}{x^2+\pi^2}dx.$$

7 (10 pt). Show that

$$\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

by applying $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$.