Qualifying Examination - Analysis - 01/2018

1. (a) (7pts) Compute

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{\frac{1}{n}}.$$

(b) (8pts) Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{\sin n}{n}$$

2. (10pts) Let $f(x)$ be a continuous function on $[0,1]$. Show that

$$\lim_{p\to\infty}\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} = \sup_{x\in[0,1]}|f(x)|.$$

3. (10pts) Let $f(x)$ be a continuous function on $[0,1]$ satisfying
(1) $f(0)=0$,
(2) $\int_0^1 f(x)x^k\,dx = 0$ for any $k=1,2,3,\ldots$
Show that $f(x)=0$ for all x.

4. (10pts) Let m be the Lebesque measure on $\mathbb{R}$. Assume that $\{E_n\}_{n=1}^{\infty}$ is a countably family of measurable sets in $\mathbb{R}$ and that

$$\sum_{n=1}^{\infty} m(E_n) < 0.$$

Show that the set

$$E = \{x\in\mathbb{R} : x\in E_n \text{ for infinitely many } n\}$$

is measurable and $m(E)=0$.

5. (10pts) Solve the nonhomogeneous Bessel's equation:

$$x^2y'' + xy' + \left(x^2-\frac{1}{4}\right)y = x\sqrt{x},\ y(\pi)=1,\ y'(0)=0.$$

6. Consider the following initial-value problem:

$$y'' - 3y' + 2y = e^{3t},\ y\,(0)=1,\ y'\,(0)=0,$$

(a) (7pts) Solve the equation above without using the Laplace transform.
(b) (8pts) Solve the equation above with using the Laplace transform.

7. (10pts) Let $R>1$ and let f be holomorphic on $|z|<R$ except at $z=1$, where f has a simple pole. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z|<1)$$

is the Maclaurin series for f, show that $\lim_{n\to\infty} a_n$ exists.

8. (10pts) Evaluate $\int_0^{\infty}\frac{\sin x}{x}\,dx$.

9. (10pts) Find a conformal mapping which maps the domain D onto the open unit disc, where D is the intersection of $|z|<1$ and $|z-1|<1$.