Algebra Qualifying Exam

January 25, 2018

1. (6, 7, 7 points) Let G be a finite group and let $H \subset G$ be a subgroup of index $|G : H| = n$.

(a) Show that $|H : (H \cap Hg)| \leq n$ for all $g \in G$.

(b) If H is a maximal subgroup of G and H is abelian, show that $(H \cap Hg)$ is a normal subgroup of G for all $g \notin H$.

(c) Now suppose that G is simple. If H is abelian and n is a prime, prove that $H = 1$.

2. (15 points) List, by giving generators for them, the ideals of $\mathbb{Z}[x]/\langle 4, x^2 \rangle$.

3. (15 points) Let A, B, C be R-modules. Show that

$$\mathrm{Hom}\,(A \otimes B, C) \cong \mathrm{Hom}\,(A, \mathrm{Hom}\,(B, C))\,.$$

4. (15 points) Find all the possible Galois groups for the splitting field of $x^3 - a$ over $\mathbb{Q}$, where a is an integer.

5. (10, 10 points) Answer the following questions.

(a) Show that the nth cyclotomic polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

(b) Determine the factorization of $\Phi_7(x) = x^6 + x^5 + \cdots + x + 1$ over $\mathbb{F}_p$ where p is a prime.

6. (15 points) The special linear group $\mathrm{SL}_2(\mathbb{C})$ is defined to be the set of 2×2 matrices of determinant 1 with entries in complex numbers. Classify finite abelian subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to isomorphism.

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