Probability Theory Qualifying Examination
June 2017 POSTECH

(5 problems. Justify all your work.)

1. (20 pts)

   (a) Suppose X_n converges weakly to X and $Y_n \geq 0$ converges weakly to some constant $c > 0$. Show that $X_n Y_n$ converges weakly to cX.

   (b) Let $X_1, X_2, \dots$ be i.i.d random variables with mean 0 and variance $\sigma^2 > 0$. Show that $\frac{X_1+\cdots+X_n}{\sqrt{X_1^2+\cdots+X_n^2}}$ converges weakly to the standard normal distribution.

2. (20 pts) Let $X_1 = 0$ and let $X_2, X_3, \dots$ be independent random variables such that
$$\mathbf{P}(X_k = k) = \frac{1}{2k\log k}, \quad \mathbf{P}(X_k = -k) = \frac{1}{2k\log k}, \quad \mathbf{P}(X_k = 0) = 1 - \frac{1}{k\log k}.$$
   Let $S_n := X_1 + \cdots + X_n$. Show that $\frac{S_n}{n}$ converges to 0 in probability but it does not converge to 0 almost surely.

3. (20 pts) Suppose that $A_1, A_2, \dots$ are pairwise independent and $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. Show that
$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbf{1}_{A_m}}{\sum_{m=1}^{n} \mathbf{P}(A_m)} = 1 \quad \text{almost surely.}$$

4. (20 pts) Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, i.e., $\mathcal{F}_n$ is an increasing sequence of σ-fields and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. Let $X \in L^1$. Show that
$$\lim_{n\to\infty} \mathbf{E}(X|\mathcal{F}_n) = \mathbf{E}(X|\mathcal{F}_\infty) \quad \text{a.s. and in } L^1.$$

5. (20 pts) Let $X_1, X_2, \dots$ be nonnegative i.i.d random variables with $\mathbf{E}X_k = 1$ and $\mathbf{P}(X_k = 1) < 1$. Let $M_0 = 1$ and $M_n := X_1 X_2 \cdots X_n$. Define $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$.

   (a) Show that $M_n := X_1 X_2 \cdots X_n$ is a martingale with respect to $\mathcal{F}_n$.

   (b) Show that M_n converges to 0 almost surely as $n \to \infty$.