Qualifying Exam (Algebra I)

January 5, 2017

1.(10 points) Let p be a prime dividing the order of a finite group G. Show that G contains an element of order p.

2.(5 points each) Determine each of the following statement is true or false and justify your answer: Let S_n denote the symmetric group of degree n.

(a) S_5 has a subgroup of order 15.

(b) S_5 has a subgroup of order 20.

(c) S_5 has a subgroup of order 40.

3.(10 points each) Answer the following questions:

(a) Determine all ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z} \times \mathbb{Z}$ denotes the direct product of two copies of the ring $\mathbb{Z}$.

(b) Let $\mathbb{Z}_n$ denote the ring of integers modulo n. Express the group of automorphisms $Aut(\mathbb{Z}_{100})$ of $\mathbb{Z}_{100}$ as the direct sum of cyclic groups.

(c) Determine whether the polynomial $f(x) = x^6 + x^3 + 1$ is irreducible over the field of rational numbers $\mathbb{Q}$.

4. Let G be a group and Γ, Γ' two subgroups of G. We say that Γ and Γ' are commensurable and denote $\Gamma \cong \Gamma'$ if $[\Gamma : \Gamma \cap \Gamma'] < \infty$ and $[\Gamma' : \Gamma \cap \Gamma'] < \infty$. For a subgroup Γ of G, we set $\widetilde{\Gamma} = \{g \in G : g\Gamma g^{-1} \cong \Gamma\}$. Answer the following questions:

(a) (5 points) Show that $\widetilde{\Gamma}$ is a subgroup of G.

(b) (5 points) Show: If $\Gamma \cong \Gamma'$, then $\widetilde{\Gamma} = \widetilde{\Gamma}'$.

(c) (10 points) Suppose that $\Gamma \cong \Gamma'$. Show that, for an element α of $\widetilde{\Gamma}$, we have the coset decompositions

$$\Gamma\alpha\Gamma' = \cup_j \Gamma\alpha\gamma_j = \cup_k \delta_k \alpha\Gamma'$$

where $\cup$ denotes a disjoint union and $\{\gamma_j\}$ (resp. $\{\delta_k\}$) is a finite set of left (resp. right) coset representatives of $(\Gamma' \cap \alpha\Gamma\alpha^{-1}) \setminus \Gamma'$ (resp. $\Gamma/\Gamma \cap \alpha\Gamma'\alpha^{-1}$). Recall that $H\alpha$ (resp. αH) is called a right (resp. left) coset of H containing α.

5.(12 points) Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. Let $(1 + 3i)$ denote the principal ideal generated by $1 + 3i$. Show that the quotient ring $\mathbb{Z}[i]/(1 + 3i)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for some m and find m.

6.(13 points) Let G be a finitely generated abelian group of free rank r. Compute $G \bigotimes_{\mathbb{Z}} \mathbb{Q}$.

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