Qualifying Exam: Algebra II, Spring 2016

1. (20 points)

   (a) Let $p(x)$ be an irreducible polynomial over a field F of characteristic p. Show that there is a unique integer k and a unique irreducible separable polynomial $q(x) \in F[x]$ such that

   $$p(x) = q(x^{p^k}).$$

   (b) Find an example of a reducible polynomial $p(x) \in F[x]$ such that the above result does not hold.

2. (20 points)

   Let n be a positive integer. Let F be a field of characteristic 0 which contains a primitive n-th root of unity. Show that any degree n cyclic extension of F is of the form $F(\sqrt[n]{a})$ for some $a \in F$.

3. (20 points)

   Consider the polynomial $f(x) = x^5 - 6x + 2 \in \mathbb{Q}[x]$. Show that the Galois group $Gal(F/\mathbb{Q})$ of the splitting field F of $f(x)$ is isomorphic to the symmetric group S_5.

4. (20 points)

   Fine the rational canonical form and the Jordan canonical form of the following matrices

   $$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}.$$

5. (20 points)

   Let I be an ideal of a commutative ring R with identity 1. The radical of I, denoted $\text{Rad}(I)$, is defined as the ideal $\bigcap_P P$ where the intersection is taken over all prime ideals containing I. Prove that $Rad(I) = \{r \in R \ : \ r^n \in I \quad \text{for some } n > 0\}$.